P-adic Cohomology of the Drinfeld half-space. Wieslawa Niziol, 1/30/18.

j/w Colmez, Dospinescu

Objects of study:

P-adic/étale cohomology of symmetric spaces. Related to geometric langlands, but focus on basic computations in this talk.

Notation: p prime; $K/\mathbb{Q}_p$ finite extension; $C := \hat{\bar{K}}$

$$\mathcal{G}_K := Gal(\bar{K}/K)$$

$\mathbb{H}^d_K = \mathbb{P}^d_K \setminus \bigcup_{H \in \mathcal{H}} H$ $\quad$ $\mathcal{H}$: K-rational hyperplanes.

$G := Gl_{d+1}K$ acts on $\mathbb{H}^d_K$; rigid analytic space

Stein: $\mathbb{H}^d_K : U_n \Subset U_{n+1}$ affinoids.

facts: X - stein

basic facts hidden in all computations.

(a) acyclicity of coh. of coherent sheaves $H^i(X, \mathcal{F}) = 0$, $\mathcal{F}$ coherent, $i > 0$.

(b) $\mathfrak{X}$ formal model$/\mathcal{O}_K$ $\rightsquigarrow$ irreducible components of the special fiber are proper

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Interested in étale and proétale cohomology.

$$H^i_{\text{ét}}(X, \mathbb{Q}_p)$$
$$\downarrow \quad \text{not in general injective: e.g. open ball over } \mathbb{C}_p.$$
$$H^i_{\text{proét}}(X, \mathbb{Q}_p) \simeq H^i((\text{holim}_n R\Gamma_{\text{ét}}(U_n, \mathbb{Z}_p)) \otimes_{\mathbb{Z}} \mathbb{Q}_p)$$
$$\wr\!\!\mid$$
$$H^i(\text{holim}_i \underline{R\Gamma_{\text{proét}}(U_n, \mathbb{Q}_p)})$$

For us: $H^i_{\text{ét}}(\mathbb{H}^d_C, \mathbb{Z}_p) \circlearrowleft G$

$$H^i_{\text{proét}}(\mathbb{H}^d_C, \mathbb{Q}_p) \circlearrowleft \mathcal{G}_K$$

Theorem (Schneider-Stuhler, Iovita-Spiess, de Shalit) $0 \le r \le d$

(i) $\ell \neq p$; $\exists\, G \times \mathcal{G}_K$-equivariant isomorphisms $H^r_{\text{ét}}(\mathbb{H}^d_C, \mathbb{Q}_\ell(r)) \simeq Sp_r(\mathbb{Z}_\ell)^* \otimes \mathbb{Q}_\ell$, with G acting on $Sp_r(\mathbb{Z}_\ell)^*$

(action of $\mathcal{G}_K$ trivial)

$$H^r_{\text{proét}}(\mathbb{H}^d_{C}, \mathbb{Q}_\ell(r)) \simeq Sp_r(\mathbb{Q}_\ell)^*$$

(ii) $\exists$ isom. of G-modules $H^r_{dR}(\mathbb{H}^d_K) \simeq Sp_r(K)^*$.

Review: generalized Steinberg representations.

$$\underset{\substack{\uparrow \\ \text{abelian group}}}{Sp_r(A)} := \underset{\substack{\uparrow \\ \text{locally constant functions}}}{LC}(G/P_{\{1,\dots,d-r\}}, A) \Big/ \sum_{P' \supseteq P} LC(G/P', A)$$

where the parabolic is $P_{\{1,\dots,s\}} := \left\{ \begin{bmatrix} * & * \\ 0 & * \end{bmatrix} \right\}$ ($s+1$, $d+1$)

Facts: ① $Sp_r(A)$ is smooth G-module

② If A is a field of characteristic $=0$ or p, $Sp_r(A)$ is irreducible (classical, due to Grosse-Klönne).

★ If we have a field of char $\ell \neq p$, this fails (Vignéras).

③ More generally, $Sp_J(A)$, $J \subset \{1, \dots, d+1\}$ is irreducible

④ $Sp_J(\mathcal{O}_K)$ is, up to a $K^\times$-homothety, the unique G-stable lattice in $Sp_J(K)$.

Theorem. (Colmez - Dospinescu - Niziol). $0 \leq r \leq d$

(i) $\exists$ exact sequence of $G \times \mathcal{G}_K$ Fréchet spaces (limits of fin. sequences of Banach spaces)

$$0 \to \underbrace{\Omega^{r-1}(\mathbb{H}^d_C)/\ker d}_{\text{de Rham part}} \to H^r_{\text{proét}}(\mathbb{H}^d_C, \mathbb{Q}_p(r)) \to \underbrace{Sp_r(\mathbb{Q}_p)^*}_{\text{finite part}} \to 0$$

(ii) $\exists$ isom of $G \times \mathcal{G}_K$-modules $H^r_{\text{ét}}(\mathbb{H}^d_C, \mathbb{Q}_p(r)) \simeq Sp_r(\mathbb{Z}_p) \otimes \mathbb{Q}_p$

(iii) — " — $H^r_{\text{ét}}(\mathbb{H}^d_C, \mathbb{Z}_p(r)) \simeq Sp_r(\mathbb{Z}_p)^*$

Rmk. 1. $d=1$: Drinfeld, Fresnel - van der Put:

Kummer theory + vanishing of Picard groups for a standard Stein covering $\{U_n\}$

Rmk 2. Proof uses p-adic hodge theory: for (i), (ii)

Comparison results of Tsuji to pass from étale cohom. to crystalline or some other.

$X := \mathbb{H}^d_c$ $\qquad \mathfrak{X}_0 \overset{i}{\hookrightarrow} \mathfrak{X} \overset{j}{\hookleftarrow} X \qquad R\psi := i^* Rj_*$.

$\mathfrak{X}$ a stable model $\qquad$ special fib.

$$\tau_{\leq r} S_n(r) \overset{\tau_{\leq r}}{\simeq} R\psi\, \mathbb{Z}/p^n(r), \quad r \geq 0$$

up to $N(r)$

"Frobenius filtered eigenspaces of crystalline cohomology"

· for (iii): Application of Bhatt-Morrow-Scholze theory (need semi-stable version).

sketch: $\widetilde{\mathbb{H}}^d_K$ denotes the standard semistable formal model of $\mathbb{H}^d_K$.

$\mathfrak{X} := \widetilde{\mathbb{H}}^d_K$, $X := \mathbb{H}^d_K$. $\quad$! all differentials are log arithmetic !

Input 1: Grosse-Klönne

Thm: The model $\mathfrak{X}$ is strongly ordinary ($\Rightarrow$ $\mathfrak{X}$ is pro-ordinary.)

(reasonably simple p-adic cohom)

$H^i(\mathfrak{X}, \Omega^j) = 0, \; j \geq 0, \; i > 0$

$H^0(\mathfrak{X}, \Omega^j)$ killed by deRham differentials.

This is proved by reducing to acyclicity of coefficient systems on partial BT-buildings)

Bruhat-Tits

Input 2: Thm (CDN) [used in proof of (ii)]

$$\begin{tikzcd} H^r_{dR}(\mathfrak{X}) \simeq H^0(\mathfrak{X}, \Omega^r) & & Sp_r(\mathcal{O}_K)^* \arrow[ll, "r_{HT}"', "\sim"] \arrow[ll, bend left, "r_{dR}"] \end{tikzcd}$$

commutes.

Pf. 1 rational case. $H^r_{dR}(X) \overset{\alpha_s}{\simeq} Sp_r(K)^*$

Iovita-Spiess

$$H^r_{dR}(X) \underset{\alpha_s}{\simeq} Sp_r(K)^*$$

recall: $Sp_r(\mathbb{Z}) \simeq LC(\mathbb{H}^{r+1}, \mathbb{Z})$ "sums of $d+1$-tuples of $(H_0, H_1, \ldots, H_r)$".

locally constant distributions

$$\begin{tikzcd} 0 \arrow[r] & D(\mathbb{H}^{r+1},K)_{deg} \arrow[r] & D(\mathcal{H}^{r+1},K) \arrow[r, "r_{dR}"] \arrow[dr, "\text{commutes}"'] \arrow[d, phantom, "\shortparallel"] & H^r_{dR}(X) \arrow[d, phantom, "\wr"] \\ & & \mathrm{Hom}(LC(\mathcal{H}^{r+1},\mathbb{Z}),K) & Sp_r(K)^* \end{tikzcd}$$

} exact

$$r_{dR}: \quad \mu \longmapsto \int_{\mathcal{H}^{r+1}} \omega_{(H_0, H_1, \ldots, H_r)} \, \mu(H_0, \ldots, H_r)$$

② over $\mathcal{O}_K$: need representation theory.

Rmk: have similar computations for other differential cohomologies

Hyodo-Kato cohom. $\longrightarrow H^r(\mathfrak{X}_0, W\Omega^{\cdot}_{\mathfrak{X}_0/\mathcal{O}_F^0}) \simeq Sp_r(\mathcal{O}_F)^*$ F⊂K – abs. unramified subfield

$$H^r_{\acute{e}t}(\mathfrak{X}_0, W\Omega^r_{\log}) \simeq H^r_{\acute{e}t}(\bar{\mathfrak{X}}_0, W\Omega^r_{\log}) \simeq Sp_r(\mathbb{Z}_p)^*$$

$\wr\wr$

So computation shows $\exists \; H^r_{\acute{e}t}(X_r, \mathbb{Z}_p(r)) \simeq H^0_{\acute{e}t}(\bar{\mathfrak{X}}_0, W\Omega^r_{\log})$

Use of Bhatt-Morrow-Scholze:

First change notation. $\mathfrak{X} := \mathfrak{X}_{\mathcal{O}_C}$, $X := X_C$.

Starting point: Artin-Schreier theory

$$A_{\inf} = W(\mathcal{O}_C^\flat) \overset{\theta}{\twoheadrightarrow} \mathcal{O}_C \qquad \mathcal{O}_C^\flat := \varprojlim \mathcal{O}_C/p$$

AS sequence: $0 \longrightarrow \hat{\mathbb{Z}}_p \longrightarrow A_{\inf} \xrightarrow{1-\varphi} A_{\inf} \longrightarrow 0$

$$R\nu_* \hat{\mathbb{Z}}_p \simeq (R\nu_* A_{\inf})^{\varphi=1}$$

<u>Twisted AS</u>: $r \geq 0$

$$0 \longrightarrow \hat{\mathbb{Z}}_p(r) \longrightarrow A_{\inf}\{r\} \xrightarrow{1-\varphi^{-1}} A_{\inf}\{r\} \longrightarrow 0$$

$$\Rightarrow \quad R\nu_* \hat{\mathbb{Z}}_p(r) \xrightarrow{\sim} (R\nu_* A_{\inf}\{r\})^{\varphi^{-1}=1}$$

A_{inf} - cohom (BMS 1, 2, CK). Comparison theorems.

$$A\Omega_{\mathcal{X}} := L\eta_\mu R\nu_* A_{inf} \in D^{\geq 0}(\mathcal{X}_{et}, A_{inf})$$

comparison $\begin{cases} \text{DR}: & A\Omega_{\mathcal{X}}/\xi \simeq \Omega^{\bullet}_{\mathcal{X}/\sigma_C} \\ \text{HT}: & \mathcal{H}^r(A\Omega_{\mathcal{X}}/\xi) \simeq \Omega^r_{\mathcal{X}/\sigma_C}\{r\} \end{cases}$ $\qquad \xi := \varphi(\xi)$

<u>Thm</u> (BMS 2).

$$\tau_{\leq r}(\tau_{\leq r} A\Omega_{\mathcal{X}}\{r\})^{\varphi^{-1}=1} \simeq \tau_{\leq r} R\nu_* \hat{\mathbb{Z}}_p(r)$$

<u>Have</u>: $0 \longrightarrow \underbrace{H^{r-1}_{et}(\mathcal{X}, A\Omega\{r\})}_{\text{show vanishes}} / (1-\varphi^{-1}) \longrightarrow H^r_{et}(X, \mathbb{Z}_p(r)) \longrightarrow \underbrace{H^r_{et}(\mathcal{X}, A\Omega\{r\})^{\varphi^{-1}}}_{\text{show gives you what you want}} \to 0$

<u>Use</u>:

<u>Thm</u> (CDN). $\exists$ natural

$$r_{inf}: A_{inf} \hat{\otimes} Sp_r(\mathbb{Z}_p)^* \xrightarrow{\sim} H^r_{et}(\mathcal{X}, A\Omega\{r\})$$

compatible with r_{dR}.