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Speaker's Name: Kathryn Hess

Talk Title: Topological Hochschild homology and topological cyclic homology: from classical to modern - III

Date: 2 / 8 / 2019 **Time:** 9 : 30 am / pm (circle one)

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THH AND TC: FROM CLASSICAL TO MODERN - III

KATHRYN HESS

1. Cyclotomic spectra

Definition 1.1. We define the category of *orthogonal cyclotomic spectra* CycSp^O:

- Objects are pairs $(X, (\varphi_n)_{n\geq 0})$ where
  - $X \in \mathbf{T}\mathrm{Sp}^O$,
  - $\varphi_n\colon \Phi^{C_n}(X) \xrightarrow{\sim} X \in \mathbf{T}\mathrm{Sp}^O$ where the equivalence is in the sense of $\mathcal{F}$-equivalence. (Note that $\Phi^{C_n}(X)$ is still a $\mathbf{T}$-spectrum via the identification $\mathbf{T}/C_n \xrightarrow{\sim} \mathbf{T}$.)
  - We demand that the diagram commutes:

$$\begin{array}{ccc} \Phi^{C_n}\Phi^{C_m}(X) & \xrightarrow{\sim} & \Phi^{C_{nm}}(X) \\ \big\downarrow & & \big\downarrow{\scriptstyle \varphi_{nm}} \\ \Phi^{C_n}(X) & \xrightarrow{\varphi_n} & X \end{array}$$

  Remark 1.2. These structures should be thought of as analogous to the isomorphism $(\mathscr{L}X)^{C_n} \cong \mathscr{L}X$.

- A map $f \in \mathrm{CycSp}^O((X, (\varphi_n)_{n\geq 0}), (Y, (\psi_n)_{n\geq 0}))$ is an *$\mathcal{F}$-equivalence* if it is on underlying $\mathbf{T}$-spectra.

We can now define topological cyclic homology for any orthogonal spectrum.

Definition 1.3. Given $(X, (\varphi_n)_{n\geq 0}) \in \mathrm{CycSp}^O$, we define its *topological cyclic homology* $\mathrm{TC}(X)$ as in the first lecture, using

$$F\colon X^{C_{p^n}} \to X^{C_{p^{n-1}}}$$

$$R\colon X^{C_{p^n}} \to (\Phi^{C_p}X)^{C_{p^{n-1}}} \xrightarrow{\varphi_p^{C_{p^{n-1}}}} X^{C_{p^{n-1}}}$$

1.1. Cyclotomic structure on THH.

Let $A \in \mathrm{Alg}(\mathrm{Sp}^O)$. We need a replacement for $A \wedge \ldots \wedge A$ to get homotopy invariance and to get the cyclotomic structure. (Recall we said before this would only behave well if A was "cofibrant".)

[Bökstedt] gave a construction $B(A, \ldots, A)$ such that there is a natural equivalence

$$A \wedge \ldots \wedge A \xrightarrow{\sim} B(A, \ldots, A)$$

which preserves equivalences of Sp^O, and plays nicely with geometric fixed points.

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Thanks to this, given $A \in \mathrm{Alg}(\mathrm{Sp}^O)$, Bökstedt's $\mathrm{THH}_*(A)$ has

$$\mathrm{THH}_n(A) := B(A, \dots, A)_{n+1}.$$

Then we define

$$|\,\mathrm{THH}(A)| := |\,\mathrm{THH}_*(A)|.$$

Properties:

(1) There is a natural cyclotomic structure on $\mathrm{THH}(A)$.
(2) There is a natural trace map $\mathrm{tr}\colon K(A) \to \mathrm{THH}(A)$.
(3) (Bökstedt-Waldhausen) $\mathrm{THH}(\Sigma^\infty_+(\Omega X)) \cong \Sigma^\infty_+(\mathscr{L}X)$.
(4) THH generalizes to $\mathrm{Sp}^O - \mathrm{Cat}$ and $\mathrm{THH}(\mathrm{Perf}_A) \cong \mathrm{THH}(A)$.
(5) Morita invariance and localization generalize.

2. Cyclotomic spectra, re-imagined

We are going to move from the world of model categories and ∞-categories.

General principle [Lurie]: there is a functor ("homotopy coherent nerve") $N\colon \mathrm{sCat} \to \mathrm{sSet}$ such that if $\mathcal{C}(x, y)$ is a Kan complex for all $x, y \in \mathcal{C}$, then $N\mathcal{C}$ is a quasi-category.

Remark 2.1. Kan complexes are the fibrant objects in the model category structure on simplicial sets.

If $\mathcal{M}$ is a simplicial model category, then $N\mathcal{M}_{cf}$ is a quasi-category, which is referred to as the "underlying ∞-category of $\mathcal{M}$". (Here $\mathcal{M}_{cf}$ is the subcategory of cofibrant and fibrant objects.)

Notation: for any simplicial model category $\mathcal{M}$, we write $N\mathcal{M}$ for this ∞-category.

A result of Hinich, generalized by Nikolas-Scholze, asserts that if $\mathcal{M}$ has a symmetric monoidal structure then $N\mathcal{M}$ inherits a symmetric monoidal structure. More precisely, if $(\mathcal{M}, \otimes, \mathbf{I})$ is a symmetric monoidal model category then $N\mathcal{M}$ is a symmetric monoidal ∞-category.

2.1. Dictionary.

Model Category	∞-category	$C_{p^\infty}\mathrm{Sp} = \varprojlim_n C_{p^n}\mathrm{Sp}$
Sp^O	Sp	
GSp^O	GSp	
$\mathbf{T}\mathrm{Sp}^O_{\mathcal{F}}$	$\mathbf{T}\mathrm{Sp}_{\mathcal{F}}$	

Induced functors:

- We have a "forgetful functor" $U\colon \mathrm{GSp} \to \mathrm{Sp}^{BG}$ (since the forgetful functor $\mathrm{GSp}^O \to \mathrm{Sp}^O$ preserves equivalences).
- $\Phi^H, (-)^H\colon \mathrm{GSp} \to \mathrm{Sp}$ for all $H < G$ preserve equivalences, and there is a natural transformation

$$(-)^H \to \Phi^H(-)$$

which corresponds to "the inclusion of the trivial representation in the regular representation".

- For all $H < G$, we can consider the composite
$$\mathrm{GSp} \xrightarrow{U} \mathrm{Sp}^{BG} \xrightarrow{\lim_{BG}} \mathrm{Sp}$$
is the "homotopy fixed points" functor $(-)^{hH}$. Similarly there is a homotopy orbit functor $(-)_{hH}$. There is a natural transformation $(-)^H \to (-)^{hH}$.

What about cyclotomic spectra? There are two possible answers:

2.2. **First approach.** Fix a prime p.
$$\mathrm{CycSp}_p^{\mathrm{gen}} = \mathrm{Eq}\left(C_{p^\infty}\mathrm{Sp} \underset{\Phi^{C_p}}{\overset{\mathrm{Id}}{\rightrightarrows}} C_{p^\infty}\mathrm{Sp} \right).$$
(Here we use that $C_{p^\infty}/C_p \cong C_{p^\infty}$.) This is the ∞-category of *genuine p-cyclotomic spectra.*

What does this look like? The objects are $(X, \Phi^{C_p}(X) \xrightarrow{\sim} X)$.

For the integral case, observe that inverting $\mathcal{F}$-equivalences gives an action of $\mathbf{N}_{\geq 0}$ on $\mathbf{T}\mathrm{Sp}_{\mathcal{F}}$. We then define $\mathrm{CycSp}^{\mathrm{gen}} = (\mathbf{T}\mathrm{Sp}_{\mathcal{F}})^{h\,\mathbf{N}_{>0}}$. This is the category of *genuine cyclotomic spectra*.

The objects are $(X, (\varphi_n \colon \Phi^{C_n}X \xrightarrow{\sim} X)_{n>0})$ with the φ_n being $\mathbf{T}$-equivariant, plus a homotopy-coherent commutativity condition. (This is one reason we'll look at a second model for cyclotomic spectra.)

Theorem 2.2 (Nikolaus-Scholze)**.** $\mathrm{CycSp}^{\mathrm{gen}}$ *"is" the underlying ∞-category of* CycSp^O*. (There is a similar result for a fixed p.)*

2.3. **Nikolaus-Scholze approach.** We first need to generalize the classical Tate construction.

Definition 2.3. Let G be a finite group. Let $\mathcal{C}$ be a stable ∞-category admitting all limits and colimits indexed by BG. It turns out that there is a norm map
$$\mathrm{Nm}_G \colon (-)_{hG} \to (-)^{hG}.$$
The *Tate construction* is the functor $(-)^{tG} \colon \mathcal{C}^{BG} \to \mathcal{C}$ takes X to the cofiber of $\mathrm{Nm}_G \colon X_{hG} \to X^{hG}$.

Example 2.4. Let $\mathcal{C} = \mathrm{Sp}$. For a G-module M, let HM be the corresponding Eilenberg-MacLane spectrum. Then
$$\pi_i(M^{tG}) \cong \widehat{H}^{-i}(G; M)$$
(the classical Tate construction).

Definition 2.5. The ∞-category of (Nikolaus-Scholze) p-cyclotomic spectra is
$$\begin{array}{ccc} \mathrm{CycSp}_p & \longrightarrow & (\mathrm{Sp}^{BC_{p^\infty}})^{\Delta^1} \\ \downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0, \mathrm{ev}_1} \\ \mathrm{Sp}^{BC_{p^\infty}} & \xrightarrow{(\mathrm{Id}, (-)^{tC_p})} & \mathrm{Sp}^{BC_{p^\infty}} \times \mathrm{Sp}^{BC_{p^\infty}} \end{array}$$

So the objects can be thought of as pairs $(X \in \mathrm{Sp}^{BC_{p^\infty}}, \varphi_p\colon X \to X^{tC_p})$ with φ_p being C_{p^∞}-equivariant.

What about the integral case? We define CycSp by the limit of the diagram:

$$\begin{array}{ccc}
\mathrm{CycSp} & \longrightarrow & \left(\prod_{p \text{ prime}} \mathrm{Sp}^{B\mathbf{T}}\right)^{\Delta^1} \\
\downarrow & & \downarrow{\scriptstyle \mathrm{ev}_0, \mathrm{ev}_1} \\
\mathrm{Sp}^{B\mathbf{T}} & \xrightarrow{(\Delta, ((-)^{tC_p})_p)} & \left(\prod_{p \text{ prime}} \mathrm{Sp}^{B\mathbf{T}}\right)^2
\end{array}$$

The objects are $(X \in \mathrm{Sp}^{B\mathbf{T}}, (\varphi_p\colon X \to X^{tC_p})_p)$ where the φ_p are $\mathbf{T}$-equivariant.

Example 2.6. We make the sphere spectrum into a cyclotomic spectrum. We have $\mathbf{S}^{\mathrm{triv}} \in \mathrm{Sp}^{B\mathbf{T}}$. Since the action is trivial, there is a map $\mathbf{S} \to \mathbf{S}^{hC_p}$ and the composite map to $\mathbf{S}^{tC_p}$ is defined to be φ_p. One has to show that this composite is actually $\mathbf{T}$-equivariant.

Theorem 2.7 (Nikolaus-Scholze)**.** *There are equivalences of ∞-categories*

$$\mathrm{CycSp}^{\mathrm{gen}} \xrightarrow{\sim} \mathrm{CycSp}$$

and

$$\mathrm{CycSp}_p^{\mathrm{gen}} \xrightarrow{\sim} \mathrm{CycSp}_p$$

when restricted to bounded below spectra.

3. TC for Nikolaus-Scholze cyclotomic spectra

Remark 3.1. Let $\mathcal{C}$ be a stable ∞-category, so for all $x, y \in \mathrm{Ob}(\mathcal{C})$ there is a mapping space $\mathrm{Map}_{\mathcal{C}}(x, y) \in \mathrm{Sp}$.

Definition 3.2. For $X \in \mathrm{CycSp}$, we define $\mathrm{TC}(X)$ to be $\mathrm{Map}_{\mathrm{CycSp}}(\mathbf{S}^{\mathrm{triv}}, X)$. (Similarly for p.)

Theorem 3.3 (Nikolaus-Scholze)**.** *If $X \in \mathrm{CycSp}^{\mathrm{gen}}$ is bounded below, then*

$$\mathrm{TC}^{\mathrm{gen}}(X) \cong \mathrm{TC}(X)$$

and similarly for p.

Theorem 3.4. *Let $A \in \mathrm{Alg}_{\mathbf{E}_1}(\mathrm{Sp})$, intuitively an "associative spectrum", then there exists a natural cyclotomic structure on* $\mathrm{THH}(A)$.