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Talk Title: DAG I: the cotangent complex and derived de Rham cohomology

Date: 01/31/19  Time: 9:30 am/ pm (circle one)

Please summarize the lecture in 5 or fewer sentences:

Introduction to DAG

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1. Motivation

We are going to give an evolutionary diagram.

- In classical algebraic geometry, one considers “algebraic schemes” or “algebraic spaces” such as $\mathbb{P}^n$.
- Then one considers “algebraic stacks” such as the Picard stack $\text{Pic}_X/k$ of a curve $X/k$. This is an “Artin stack”.

To make the next step, we need to adopt a new point of view. We can view these objects as sheaves of sets (by the functor of points). We then view sets as “0-truncated spaces” $S_{\leq 0}$, i.e. spaces with non-zero homotopy groups $\pi_i$ with $i \leq 0$.

More generally, we can consider groupoids, which are “1-truncated spaces” $S_{\leq 1}$.

If we continue, we enter the realm of “higher stacks”, e.g. $K(\mathbb{G}_m, n)$. The isotopy of this space is $K(\mathbb{G}_m, n-1)$.

**Example 1.1.** We can reformulate $\text{Pic}_X/k$ as the mapping stack from $X$ to $K(\mathbb{G}_m, 1)$, which also goes by the name $BG_m = [\text{pt} / \mathbb{G}_m]$. We can then view $K(\mathbb{G}_m, 2) = [\text{pt} / BG_m]$.

**Example 1.2.** What does it mean to give a map from a scheme to $K(\mathbb{G}_m, n)$? By definition $\text{Map}(X, K(\mathbb{G}_m, n))$ is a topological space, with

$$\pi_i \text{Map}(X, K(\mathbb{G}_m, n)) \cong \begin{cases} H_{\text{et}}^{n-i}(X, \mathbb{G}_m) & 0 \leq i \leq n, \\ 0 & \text{otherwise.} \end{cases}$$

We have now expanded our world of “geometric objects” to include sheaves of groupoids, or sheaves of $n$-truncated spaces. The fundamental idea of derived algebraic geometry is to allow the sheaf of functions itself to be a sheaf of topological spaces. This leads to the notion of “derived schemes”.

**Example 1.3.** An example of an affine derived scheme is $\text{Spec}(k \otimes^L_{k[x]} k)$, where the maps $k[x] \to k$ send $x \mapsto 0$.

2. Simplicial commutative rings

We are now going to introduce a model for what should be the “commutative rings of derived algebraic geometry”.

*Date: January 31, 2019.*
Consider the derived category \( D(\mathbb{Z})_{\geq 0} \), where we are considering only the complexes \( C_\ast \) such that \( H_i(C_\ast) = 0 \) for \( i < 0 \) (this condition is also called “connective”). This can be enhanced to a symmetric monoidal category using the derived tensor product \( \otimes^L \).

What should we take as the commutative algebras in \( (D(\mathbb{Z})_{\geq 0}, \otimes^L) \)? There are three different answers:

- \( E_\infty \)-ring spectra,
- simplicial commutative rings,
- Over \( \mathbb{Q} \), one can work with \( \mathbb{Q} \)-CDGAs.

These are not equivalent in general, although they are equivalent rationally. We will work with the second.

We introduce the simplex category \( \Delta \), which is the category of non-empty finite ordered sets, with morphisms being order-preserving maps of sets.

**Example 2.1.** We write \([0]\) for \( \{0\} \), \([1]\) for \( \{0,1\} \), and \([2]\) for \( \{0,1,2\} \). Evidently we have two maps \([0] \to [1]\) and one map \([1] \to 0\), etc.

**Definition 2.2.** Let \( C \) be a category. We let \( sC = \text{Fun}(\Delta^{op}, C) \) be the category of simplicial objects in \( C \).

**Example 2.3.** There is an “equivalence” between \( s\text{Sets} \) and the category of topological spaces, at the level of homotopy categories. This means we specify a notion of “weak equivalence” on each side (in topological spaces it is the usual notion, wherein maps inducing isomorphisms of \( \pi_\ast \) are weak equivalences), and the “localization” of each side with respect to these are equivalent.

Let \( \Delta^n \) be the pre-sheaf \( \text{Hom}_\Delta(-,[n]) \) i.e. the Yoneda embedding of \([n]\). Let \( \Delta^n_{top} \) be the usual \( n \)-simplex. This induces a functor from \( s\text{Sets} \) to topological spaces, called geometric realization, giving by presenting a simplicial set as a colimit of the representable objects \( \Delta^n \) and taking that colimit in \( \text{Top} \). We then pull back the notion of weak equivalence along this functor.

We can view \( \Delta^\ast_{top} \) as a cosimplicial object in \( \text{Top} \). Hence \( \text{Hom}_{\text{top}}(\Delta^\ast_{top}, X) \) is a simplicial object in \( \text{Top} \). This is the right adjoint to geometric realization.

Let’s make a connection to something familiar. Given a topological space \( X \), we make a simplicial set \( \text{Sing}(X) := \text{Hom}_{\text{top}}(\Delta^\ast_{top}, X) \). Then we form a simplicial abelian group \( \mathbb{Z}[\text{Sing}(X)] \). We can extract from this a chain complex \( C_\ast(\mathbb{Z}[\text{Sing}(X)]) \), and by definition

\[
H_i(C_\ast(\mathbb{Z}[\text{Sing}(X)])) \cong H_i^{\text{sing}}(X;\mathbb{Z}).
\]

**Example 2.4.** The Dold-Kan correspondence furnishes an equivalence \( s\text{Ab} \cong D(\mathbb{Z})_{\geq 0} \).

Given a simplicial abelian group

\[
M_0 \leftarrow M_1 \ldots
\]

we make a chain complex with differentials \( \sum (-1)^i d_i \):

\[
M_0 \xleftarrow{d_0-d_1} M_1 \leftarrow \ldots
\]
Example 2.5. The category of simplicial commutative rings is
\[ \text{sCAlg}_k = \text{Fun}(\Delta^{op}, \text{CAlg}_k). \]

There is a notion of weak equivalence, which is weak equivalence of the underlying simplicial sets. But moreover, there is a model category structure, which specifies weak equivalences but also specifies a “right way” to perform certain derived operations. For this reason, this is sometimes called a “non-abelian derived category”.

For \( R \in \text{sCAlg}_k \), \( \pi_* R \) has the structure of a graded commutative ring. This means that
\[ xy = (-1)^{|x||y|}yx \]
and \( x^2 = 0 \) if \( |x| \) is odd.

There is an adjunction \( \text{Sets} \to \text{CAlg}_k \) sending \( S \mapsto k[S] \). Given \( R \in \text{CAlg}_k \), we can make a simplicial \( k \)-algebra
\[ S_* : \ldots \to k[[R]] \Rightarrow k[R] \]
which is the analogue of a “projective resolution” for commutative rings. Let \( \text{CAlg}^\text{poly}_k \) be the category of finitely generated polynomial \( k \)-algebras. There is a fully faithful embedding to \( \text{Ind}(\text{CAlg}^\text{poly}_k) \), which is the category obtained by formally adjoining colimits, and this includes into \( \text{sCAlg}_k \).

Given a functor \( F : \text{CAlg}^\text{poly}_k \to \mathcal{C} \), there is a way to extend to \( LF : \text{sCAlg}_k \to \mathcal{C} \). The recipe is as follows. Given \( R \in \text{CAlg}_k \), make \( S_* \xrightarrow{\sim} R \) as above. Then we have \( F(S_*) \in s\mathcal{C} \), and define
\[ LF(R) = \colim_{\Delta} F(S_*) =: |F(S_*)|. \]

3. The cotangent complex

We give several constructions.

(a) Given \( k \to R \), we have explained that we can make a simplicial a “good” simplicial “resolution” \( S_* \xrightarrow{\sim} R \). Then we define
\[ L_{R/k} := \Omega^1_{S_*/k} \otimes_{S_*} R \in s\text{Mod}_R \cong D(R)_{\geq 0}. \]

(b) We have a functor \( \Omega^1_{\mathcal{C}/k} : \text{CAlg}^\text{poly}_k \to D(k)_{\geq 0} \). We can then extend this to a \( L\Omega^1_{\mathcal{C}/k} \) on \( \text{sCAlg}_k \) on as discussed previously. However, there are several deficiencies, e.g. we don’t see the \( R \)-module structure.

(c) We’ll correct the previous issues by phrasing a universal property. Let \( k \to R \to S \) and \( M \) be an \( S \)-module in \( D(S)_{\geq 0} \). We will define \( L_{R/k} \) to have the universal property that (the topological space)
\[ \text{Map}_{\mathcal{C}}(L_{R/k}, M) \cong \text{Map}_{s\text{CAlg}_k/S}(R, S \oplus M). \]

Here \( \text{Map}_{s\text{CAlg}_k/S} \) is the slice category of simplicial commutative rings equipped with a map to \( S \), and \( S \oplus M \) is the square-zero extension of \( S \) by \( M \). Note that this makes sense for \( M \in D(S)_{\geq 0} \).
Exercise 3.1. Show that $\pi_0 L_{R/k} \cong \Omega^1_{R/k}.$

Exercise 3.2. Show that $S \otimes^L_{R} L_{R/k} \to L_{S/k} \to L_{S/R}$ is an exact triangle.

Example 3.3. Show that $T \otimes_k L_{R/k} \cong L_{R \otimes_T^L k T}$

Exercise 3.4. Let $R$ be a perfect algebra over $F_p$. Using that the Frobenius morphism $\varphi: R \to R$, which sends $x \mapsto x^p$, is an isomorphism, show that $L_{R/F_p} \cong 0$.

4. Derived de Rham cohomology

There is a functor
$$dR_{R/k}: \text{CAlg}_{k}^{\text{poly}} \to D(k)$$
sending $R$ to the chain complex $(R \to \Omega^1_{R/k} \to \Omega^2_{R/k} \to \ldots)$. We can then “derive” this functor to get a “derived de Rham cohomology” functor
$$L dR_{R/k}: s\text{CAlg}_{k}^{\text{poly}} \to D(k).$$

There is something to be cautious about. If $k$ is a $\mathbb{Q}$-algebra and $R \in \text{CAlg}_{k}^{\text{poly}}$, then $dR_{R/k} \cong k$. This implies that $L dR_{S/k} \cong k$ for any $s \in s\text{CAlg}_{k}$, and this isn’t very interesting.

A fix was given by Bhargav Bhatt, which gives the answer you “want”. We also have a Hodge filtration $F^*_H$ on $dR_{R/k}$, with
$$\text{gr}^i_H (dR_{R/k}) \cong \Omega^i_{R/k}[-i].$$

We then try to take the derived functor remembering the filtration. We then get a filtration $F^*_H L dR_{-/k}$ such that
$$\text{gr}^i_H L dR_{-/k} \cong L(\wedge^i L_{-/k}[-i]).$$

This filtration isn’t complete, so we define
$$\widehat{L dR}_{-/k} := \lim_i \frac{L dR_{-/k}}{F^*_H}.$$ 

Theorem 4.1 (Bhatt, Grothendieck, Hartshorne). Let $X/\mathcal{C}$ be finite type. Then
$$R\Gamma(X, \widehat{L dR}_{\mathcal{O}_X/k}) \cong R\Gamma_{\text{sing}}(X(\mathcal{C}); \mathcal{C}).$$