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# NOTETAKER CHECKLIST FORM

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**Speaker's Name:** Benjamin Antieau

**Talk Title:** DAG II: moduli of objects in derived categories

**Date:** 2 / 1 / 19 **Time:** 11 : 00 am / pm (circle one)

**Please summarize the lecture in 5 or fewer sentences:**

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# DAG II: MODULI OF OBJECTS IN DERIVED CATEGORIES

BEN ANTIEAU

## 1. Simplicial commutative rings

Let $k$ be a simplicial commutative ring. We will consider $s\mathrm{CAlg}_k$.
For $R, S \in s\mathrm{CAlg}_k$ the object $\mathrm{Map}_{s\mathrm{CAlg}_k}(R, S)$ is a space.

**Example 1.1.** (i) $\mathrm{Map}_{s\mathrm{CAlg}_k}(k, S)$ is contractible.
(ii) $\mathrm{Map}_{s\mathrm{CAlg}_k}(k[t], S)$ is the underlying simplicial set $S$.

We have defined a cotangent complex $L_{R/k}$.

**Definition 1.2.** We say $R \in s\mathrm{CAlg}_k$ is *discrete* if $\pi_i(R) = 0$ for $i > 0$.

We have an adjunction

$$\pi_0 \colon s\mathrm{CAlg}_k \leftrightarrow s\mathrm{CAlg}_k^{\mathrm{discrete}} \cong \mathrm{CAlg}_{\pi_0(k)}.$$

**Definition 1.3.** We say $k \to R$ is *locally finite presented*, and write $R \in s\mathrm{CAlg}_k^{\omega} \subset s\mathrm{CAlg}_k$, if

$$\mathrm{Map}_{s\mathrm{CAlg}_k}(R, -) \colon s\mathrm{CAlg}_k \to S$$

commutes with filtered colimits.

**Definition 1.4.** We say that $k \to R$ is *formally étale* if $L_{R/k} \cong 0$. We say that $k \to R$ is *étale* if it is formally étale and locally of finite presentation.

**Proposition 1.5.** *The following are equivalent:*

*(a) $k \to R$ is étale.*
*(b) $\pi_0(k) \to \pi_0(R)$ is étale and $\pi_i(k) \otimes_{\pi_0(k)} \pi_0(R) \xrightarrow{\sim} \pi_i(R)$ is an isomorphism (the map $\pi_0(k) \to \pi_0(R)$ is flat, so the tensor product does not need to be derived).*

**Theorem 1.6.** *There is an equivalence*

$$s\mathrm{CAlg}_k^{\acute{e}t} \cong \mathrm{CAlg}_{\pi_0(k)}^{\acute{e}t}.$$

**Remark 1.7.** This implies that the small étale site of $k$ agrees with the small étale set of $\pi_0(k)$. Recall the topological invariance of the small étale site: for $R$ an ordinary commutative algebra and $I \subset R$ a nilpotent ideal, $\mathrm{CAlg}_R^{\text{ét}} \xrightarrow{\sim} \mathrm{CAlg}_{R/I}^{\text{ét}}$. We can think of this theorem as telling us that the higher homotopy should be regarded as nilpotents.

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*Date*: February 1, 2019.

**Example 1.8.** We can view $k \to \pi_0(k)$ as a pro-nilpotent thickening. This map factors through

$$k \to \tau_{\leq n}(k) \to \pi_0(k)$$

where

$$\pi_i(\tau_{\leq n} k) \cong \begin{cases} \pi_i(k) & 0 \leq i \leq n \\ 0 & \text{else} \end{cases}$$

and $k \cong \varprojlim_n \tau_{\leq n} k$.

## 2. Derived affine schemes

We will define $\mathrm{dAff}_k := s\mathrm{CAlg}_k$, with the equivalence denoted $\mathrm{Spec}\ R \leftrightarrow R$.

**Definition 2.1.** We say that a family of étale morphisms $\{R \to S_i\}$ is an *étale cover* if $\{\pi_0(R) \to \pi_0(S_i)\}$ is an étale cover.

We define a subcategory $\mathrm{Shv}_{\text{ét}}(\mathrm{dAff}_k) \subset \mathrm{PShv}(\mathrm{dAff}_k) \cong \mathrm{Fun}(\mathrm{sCAlg}_k, \mathcal{S})$ as follows. Given an étale cover $R \to S$, we can form the Amitsur complex

$$S \rightrightarrows S \otimes_R S \ldots$$

a cosimplicial object. The "sheaf condition" is that

$$\mathcal{F}(R) \xrightarrow{\sim} \lim_{\Delta} \mathcal{F}(S^{\otimes * + 1}).$$

As part of the definition of sheaf, we also demand that $\mathcal{F}$ preserves finite products.

The Yoneda embedding gives a fully faithful functor

$$\mathrm{dAff}_k \to \mathrm{Shv}_{\text{ét}}(\mathrm{dAff}_k).$$

Giving a simplicial commutative ring $R$, we can view $D(R) \cong \mathrm{Mod}_R(D(\mathbf{Z}))$.

**Definition 2.2.** (i) $P$ is *perfect* iff it is compact, i.e. $P \in D(R)^\omega$.

(ii) $P$ has *Tor-amplitude in* $[a, b]$ if $P \otimes_R \pi_0(R)$ has Tor-amplitude in $[a, b]$, i.e.

$$H_i(P \otimes^L_R \pi_0(R) \otimes^L_{\pi_0(R)} M) = 0 \text{ for all } i \notin [a, b].$$

**Definition 2.3.** We say $R \to S$ is *smooth* if it is locally finitely presented and $L_{S/R}$ has Tor-amplitude in $[0, 0]$.

## 3. Deried stacks

We say $X \xrightarrow{i} Y$ in $\mathrm{Shv}_{\text{ét}}(\mathrm{dAff}_k)$ is 0-*geometric* if for any $\mathrm{Spec}\ R \to Y$, the fibered product is a disjoint union of affine derived schemes:

$$\begin{array}{ccc} \coprod_{i \in I} \mathrm{Spec}\ S_i \cong P & \longrightarrow & \mathrm{Spec}\ R \\ \downarrow & & \downarrow \\ X & \longrightarrow & Y \end{array}$$

A 0-geometric morphism is

- *lfp* if the $S_i$ are lfp over $R$,
- *smooth* if the $S_i$ are smooth over $R$.

**Definition 3.1.** $X \to Y$ is *$n$-geometric* if for all Spec $R \to Y$, the fibered product admits a smooth surjective $(n-1)$-geometric map from $\coprod \operatorname{Spec} S_i$. (Surjective means that points lift étale-locally.)

$$\begin{array}{ccccc} \coprod_{i\in I} \operatorname{Spec} S_i & & & & \\ & \searrow\!\!\!\!\searrow & & & \\ & & P & \longrightarrow & \operatorname{Spec} R \\ & & \downarrow & & \downarrow \\ & & X & \longrightarrow & Y \end{array}$$

**Example 3.2.** We have $\mathrm{GL}_1 = \operatorname{Spec} k[t^{\pm 1}]$. Hence

$$\mathrm{GL}_1(R) = \mathrm{Map}_{\mathrm{sCAlg}_k}(k[t^{\pm 1}], R)$$

and

$$\pi_i(\mathrm{GL}_1(R)) \cong \begin{cases} \pi_0 R^\times & i = 0 \\ \pi_i(R) & i > 1 \end{cases}$$

In particular this is not valued in groupoids!

Clearly $R \mapsto \mathrm{GL}_1(R)$ is 0-geometric (it is representable).

**Example 3.3.** The sheaf $\mathbf{G}_m(R) := \pi_0(R^\times)$ is not $n$-geometric for any $n$.

**Example 3.4.** The classifying (derived) stack of $\mathrm{GL}_1$ is $[\mathrm{pt} / \mathrm{GL}_1]$. For Spec $k \to$ $\mathrm{BGL}_1$, the fibered product is

$$\begin{array}{ccc} \mathrm{GL}_1 & \longrightarrow & \operatorname{Spec} k \\ \downarrow & & \downarrow \\ \operatorname{Spec} k & \longrightarrow & \mathrm{BGL}_1 \end{array}$$

So $\mathrm{BGL}_1$ is 1-geometric.

**Example 3.5.** Next $B^2 \mathrm{GL}_1 := [\mathrm{pt} / \mathrm{BGL}_1]$ is the classifying stack of $\mathrm{BGL}_1$. (It can be thought of as the sheafification of $R \mapsto B(\mathrm{BGL}_1(R))$.) This has the property that

$$\begin{array}{ccccc} * & & & & \\ & \searrow & & & \\ & & \mathrm{BGL}_1 & \longrightarrow & \operatorname{Spec} k \\ & & \downarrow & & \downarrow \\ & & \operatorname{Spec} k & \longrightarrow & B^2 \mathrm{GL}_1 \end{array}$$

This is 2-geometric.

**Definition 3.6.** A *geometric* defined stack $M$ is one of the form

$$M = \varinjlim_{\mathbf{N}} M_i$$

where each $M_i$ is $n$-geometric for some $n$, and $M_i \to M_j$ is a monomorphism, meaning $M_i(R) \hookrightarrow M_j(R)$ is an inclusion of connected components.