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Speaker's Name: Janos Kollar

Talk Title: Moduli of canonical models

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Moduli spaces of algebraic varieties II

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Deligne–Mumford compactification $\bar{M}_g$

Stable curves:

Projective, connected, reduced curves C such that:

Local: at worst nodes: $(xy = 0)$ (locally analytically)

Global: ω_C is ample.

What is ω_C?

– smooth curve: $\omega_C = \Omega_C = T^*_C = \mathcal{O}_C(K_C)$.

– for any plane curve, Poincaré residue map

$$\Re : \omega_{\mathbb{P}^2}(C)|_C \cong \omega_C$$

– if $C = \cup_i C_i$ and $P_i \subset C_i$ are the nodes then

$$\omega_C|_{C_i} = \omega_{C_i}(P_i).$$

Higher dimension, basic questions

What are the correct analogs of
smooth, projective curves of genus ≥ 2?

What are the correct analogs of stable curves?

Usually:
EASY: make it work for an open moduli space.
HARD: make it work for a compact moduli space.

Canonical models 0

X^n smooth projective variety, floating around.

To get our hands on it, want an embedding $X \hookrightarrow \mathbb{P}^N$.

This needs a holomorphic line bundle L on X with sections $s_0, \ldots, s_N \in H^0(X, L)$.

Meta claim I

The only vector bundle one can write down on a manifold/variety is the tangent bundle T_X
(and its descendants)

Meta Corollary: The only line bundles are
$\omega_X = \Omega_X^n = (\det T_X)^*$ (and its powers).

Example 1: (Franchetta conjecture) If $C \mapsto L_C$ is holomorphic then $L_C = \omega_C^m$ for some m.
(Harer, Arbarello-Cornalba, Mestrano, Kouvidakis)

Example 2: X smooth, L sufficiently ample.
The only holomorphic
$H^0(X, L) \ni s \mapsto (\text{line bundle on } (s = 0))$ is:
restrict a line bundle from X to $(s = 0)$. (M. Woolf)

Example 3: (Babylonian towers) The only vector bundles on $\mathbb{P}^\infty$ are sums of line bundles. (Tyurin, Barth)

Non-example: If X is projective and ω_X is the only line bundle then ω_X is ample, so minimal model program says $X = X$.

Meta claim II: On an interesting variety there are many line bundles, but we have to work hard to find them.

Canonical models 1

Take any ω_X^m for $m \geq 1$.
Take any basis $s_0, \ldots, s_{N(m)} \in H^0(X, \omega_X^m)$.
Get a map $\phi_m : X \dashrightarrow X_m \subset \mathbb{P}^{N(m)}$.

Theorem (Canonical models)

The closed images X_m are isomorphic to each other for $1 || m$.
Get X^{can}, the canonical model of X.

- $\dim X = 2$: Castelnuovo, Enriques (+ Mumford)
- $\dim X = 3$: Mori (+ Kawamata, Kollár, Reid, Shokurov)
- $\dim X \geq 4$: Hacon–McKernan
  (+ Birkar, Cascini, Corti, Shokurov)
  (+ Fujino-Mori)

ω on a singular variety I.

Recipe: (if X is normal)

Take smooth locus $X^0 \subset X$

$\omega_{X^0} = \Omega^n_{X^0} = (\det T_{X^0})^*$, then extend it to X.

Powers: $\omega_X^{[m]} :=$ extension of $\omega^m_{X^0}$.

Exercise: A holomorphic line bundle L^0 on X^0 has at most 1 extension to a holomorphic line bundle L on X, but it may have infinitely many extensions as a topological line bundle.

ω on a singular variety II.

- Hypersurfaces: $(g = 0) \subset \mathbb{C}^n$. Generator of ω:

$$(-1)^i \frac{dx_1 \wedge \cdots \wedge \widehat{dx_i} \wedge \cdots \wedge dx_n}{\partial g / \partial x_i}$$

- Quotients: $\mathbb{C}^n/(\text{finite group } G)$. Generator of ω^m:

$$(dx_1 \wedge \cdots \wedge dx_n)^{\otimes m}$$

  where $m = |G/G \cap \mathrm{SL}_n|$.

Canonical models: internal definition

Canonical singularity: One can pull back pluricanonical forms: $p : Y \to X$ resolution, then we have

$$p^*\omega_X^{[m]} \to \omega_Y^{[m]}.$$

Canonical model: normal, projective,

- X has canonical singularities and
- ω_X is ample.

Moduli and compactification: using GIT directly

Mumford (1965): M_g using GIT,

Mumford, Gieseker (1974-80) $\bar{M}_g$ using GIT.

Gieseker (1977): moduli of surfaces using GIT, for
high enough pluricanonical embedding,

Viehweg (1989–95): higher dimensional canonical models,
with well chosen polarization,

Chenyang Xu – Xiaowei Wang (2012) GIT compactification
of the moduli of surfaces forever depends
on the pluricanonical embedding,
(both Chow and Hilbert versions).

Compactification 2: Memento Mori

Lemma. B smooth curve, $B^0 = B \setminus \{0\}$
$f^0 : Y^0 \to B^0$ a family of canonical models.

There is at most 1 extension to

$$\begin{tikzcd}
Y^0 \arrow[d, "f^0"'] \arrow[r, phantom, "\subset"] & Y \arrow[d, "f"] \\
B^0 \arrow[r, phantom, "\subset"] & B
\end{tikzcd}$$

such that

- Y has canonical singularities and
- ω_Y (rather $\omega_{Y/B}$) is ample on every fiber.

KSB approach

B smooth curve, $B^0 = B \setminus \{0\}$
$f^0 : Y^0 \to B^0$ a flat family of canonical models.

- Take any extension $f_1 : Y^1 \to B$.
- Resolve singularities $f_2 : Y^2 \to B$, write $Y_0^2 = \sum m_i D_i$.
- Take base change $C \to B$, ramification order a multiple of the m_i. Get $f_3 : Y^3 \to C$. Now $Y_0^3 = \sum D_i'$.
- Take canonical model to get $f : Y \to C$.

$$\begin{tikzcd}
Y^0 \arrow[d, "f^0"'] & Y_C^0 \arrow[l] \arrow[d, "f_C^0"'] \arrow[r, phantom, "\subset"] & Y \arrow[d, "f"] \\
B^0 & C^0 \arrow[l] \arrow[r, phantom, "\subset"] & C .
\end{tikzcd}$$

In order to apply this we need

- special Y_0 should have $\boxed{????}$ singularities

Curve case: $(xy = 0)$ is not canonical but
$(xy + t^n = 0)$ is canonical.

Needed in general case: $0 \in D \subset X$, Cartier divisor.
Assume $X \setminus D$ has canonical sings and D has $\boxed{????}$
$\Rightarrow X$ has canonical sings.

Definition: $\boxed{????} =$ semi-log-canonical.

What is semi-log-canonical?

What is a node?

What is a node?

$C = (xy = 0) \subset \mathbb{C}^2$.
generating section σ of ω_C

$$\sigma = \frac{dx}{x} \text{ on } x\text{-axis}, \qquad \sigma = -\frac{dy}{y} \text{ on } y\text{-axis}.$$

Characterizations of nodes:

Using resolutions: $p : D \to C$ then $p^*\sigma$ has only simple poles.

Using local volume: Although the local volume is

$$\frac{i}{2\pi} \int_{|x| \le 1} \frac{dx}{x} \wedge \frac{d\bar{x}}{\bar{x}} = \infty,$$

it has only logarithmic growth:

$$\frac{i}{2\pi} \int_{|x| \le 1} |x|^{\epsilon} \frac{dx}{x} \wedge \frac{d\bar{x}}{\bar{x}} < \infty \quad \text{for } \epsilon > 0.$$

What is semi-log-canonical?

Take a resolution $f : Y \to X$. Write
$K_Y = f^*K_X + J$ and $f^*D = D_Y + E$.

- $J \geq 0$ iff X has canonical singularities and
- Mumford's semi-stable reduction: may assume that all coefficients in E equal 1.

Adjunction formula: $K_{D_Y} =$

$$\big(K_Y + D_Y\big)|_{D_Y} = \big(f^*(K_X + D) + J - E\big)|_{D_Y} = f^*K_D + (J - E)|_{D_Y}$$

Suggests: $J \geq 0 \quad \Leftrightarrow \quad (J - E)|_{D_Y} \geq -1.$

Almost what we want, but no information on exceptional divisors that are disjoint from D_Y.
Convexity of the coefficients of J settles the rest.
(Shokurov, Kollár, Kawakita, de Fernex-Kollár-Xu)

Definition of semi-log-canonical

X only nodes in codimension 1. So ω_X makes sense and $\omega_X^{[m]}$ is locally free for some $m > 0$.
Using resolutions: $f : Y \to X$ with reduced exceptional divisor E, then we have pull-backs

$$f^*\left(\omega_X^{[r]}\right) \to \omega_Y^{[r]}(rE) \quad \forall r.$$

Using local volume: σ^m: generating section of $\omega_X^{[m]}$.
Although the local volume is $\int \sigma \wedge \bar{\sigma} = \infty$, it has only logarithmic growth:

$$\left(i^?\right) \cdot \int |g|^\epsilon \cdot \sigma \wedge \bar{\sigma} < \infty$$

for every g vanishing on $\operatorname{Sing} X$ and $\epsilon > 0$.

Examples of semi-log-canonical singularities

- dim=2: canonical = smooth + Du Val
  $(xy + z^n = 0, \ldots, x^2 + y^3 + z^5 = 0)$
- dim=2: log ternimal = $\mathbb{C}^2/(\text{finite group})$
- dim=n examples:

  cone over $X \subset \mathbb{P}^N$ is lc (or slc) iff X is lc (or slc)
  and $-K_X \sim rH$ for some $r \geq 0$.

  cone over $X \subset \mathbb{P}^N$ is canonical iff X is canonical
  and $-K_X \sim rH$ for some $r \geq 1$.

Stable curves $\to$ Stable varieties

X: projective, connected

Local condition: semi-log-canonical singularities
Global condition: ω_X is ample.

What are stable families?

Wrong answer:
Flat, projective morphisms with stable fibers.

Example I

Family of varieties in $\mathbb{P}^5_{\mathbf{x}} \times \mathbb{A}^2_{st}$:

$$X := \left(rank \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 + sx_4 & x_2 + tx_5 & x_3 \end{pmatrix} \leq 1 \right).$$

Claim: the following are equivalent:

- X_{st} is semi-log-canonical (in fact klt)
- $3K_{X_{st}}$ is Cartier
- either $(s, t) = (0, 0)$ or $st \neq 0$.

Being stable is not a locally closed condition.

Case 1: $st \neq 0$:

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 + sx_4 & x_2 + tx_5 & x_3 \end{pmatrix} \to \begin{pmatrix} x_0 & x_1 & x_2 \\ x_4 & x_5 & x_3 \end{pmatrix}$$

This is $\mathbb{P}^1 \times \mathbb{P}^2$, hence even smooth.

Case 2: $s = t = 0$:

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 + sx_4 & x_2 + tx_5 & x_3 \end{pmatrix} \to \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_2 & x_3 \end{pmatrix}$$

This is the 2-cone over $\mathbb{P}^1 \hookrightarrow \mathbb{P}^3$.
The singularity is locally like $\mathbb{C}^3/\frac{1}{3}(1,1,0)$:
$\mathbb{Z}/3\mathbb{Z}$ acts with $(\epsilon, \epsilon, 1)$.

Case 3: $s = 0, t \neq 0$:

$$\begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 + sx_4 & x_2 + tx_5 & x_3 \end{pmatrix} \to \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 & x_5 & x_3 \end{pmatrix}$$

This is the cone over $F_1 \hookrightarrow \mathbb{P}^4$.
F_1 is Fano but its anticanonical embedding is into $\mathbb{P}^7$.
Here $-K_{F_1}$ is not proportional to $H \cap F_1$.

Example II – with ample K

Let $Y_m \subset \mathbb{P}^6_{\mathbf{x}} \times \mathbb{C}^1_t$ be the family

$$\sum x_i^m = 0 \text{ and } rank \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 + tx_4 & x_2 + tx_5 & x_3 \end{pmatrix} \leq 1.$$

- $Y_m \to \mathbb{C}^1_t$ is flat, projective
- stable fibers for $m \geq 5$,
- $\left(K^3_{Y_{m,t}}\right)$ is not locally constant.

Stable families

Theorem (K. 2015) Let $X \to S$ be a flat, projective morphisms with stable fibers, S reduced. Equivalent:

1. The volume of the fibers $s \mapsto \left(K^n_{X_s}\right)$ is locally constant.
2. The plurigenera $s \mapsto h^0\left(X_s, \omega^{[m]}_{X_s}\right)$ are locally constant.
3. $\omega^{[m]}_{X/S}$ is flat and commutes with base change $\forall m$.

Definition of stable families

S any Noetherian scheme.
A morphism $f : X \to S$ is stable iff

1. f is flat, projective with stable fibers and
2. $\omega_{X/S}^{[m]}$ is flat and commutes with base change $\forall m$.

Stability is representable

– $f : X \to S$: flat family, projective of pure relative dim n
– fibers at worst nodal in codim 1.

Theorem

In characteristic 0, there is a monomorphism

$$i_S : S^{\text{stable}} \to S$$

such that, for every $g : T \to S$, the following are equiv.

1. *The pull-back $f_T : X_T \to T$ is stable.*
2. *g factors through i_S.*

Main existence theorem

Fix positive n, d.
There is a projective coarse moduli space $\bar{M}_{n,d}$ parametrizing stable varieties X of dimension n such that $(K_X^n) = d$.

- moduli properties as good as for $\bar{M}_g$,
- as a scheme, much more complicated.

(Note: Proofs are complete in characteristic 0 only.)

History of the proof

Surfaces:
– (background) MMP for 3-folds, Mori
– (existence) K–Shepherd-Barron
– (finite type) Alexeev
– (projectivity) K
– (local structure) arbitrarily bad, Vakil

Higher dimensions
– (background) MMP: Hacon–McKernan $+$ H–M–Xu
– (existence) K
– (finite type) Karu, Hacon–McKernan–Xu
– (projectivity) Fujino, Kovács–Patakfalvi