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Speaker's Name: Sam Raskin

Talk Title: The notion of singular support in DAG and its applications II

Date: 2 / 5 / 19 **Time:** 2 : 00 am / [✓ pm] (circle one)

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THE NOTION OF SINGULAR SUPPORT IN DAG AND ITS APPLICATIONS II

SAM RASKIN

1. Generalizations of discussion from last time

Last time we explained that for a hypersurface $X = \{f = 0\}$ in a smooth Y, and $\mathcal{F} \in \mathrm{QCoh}(X)$ or $\mathcal{F} \in \mathrm{IndCoh}(X)$, we have an operator $\eta \colon \mathcal{F} \to \mathcal{F}[2]$. Furthermore, we had the characterization: $\mathcal{F} \in \mathrm{QCoh}(X) \subset \mathrm{IndCoh}(X)$ if and only if η is locally nilpotent, i.e. $\underrightarrow{\mathrm{colim}}\, \mathcal{F}[2n] = 0$.

We will now consider a generalization to the case where X is the vanishing locus of several functions $f_1, \ldots, f_r$ in a smooth ambient Y. A generalization of the results from last time: there are natural

$$\eta_i \colon \mathcal{F} \to \mathcal{F}[2]$$

The η_i's commute in some sense, and $\mathcal{F} \in \mathrm{QCoh}(X)$ if and only if each η_i acts locally nilpotently.

Let $\mathcal{N} \subset \mathbf{P}^{r-1}$ be a closed subvariety. We then get a subcategory $\mathrm{IndCoh}_{\mathcal{N}}(X) \subset \mathrm{IndCoh}(X)$ as follows. Let $I \subset k[\eta_1, \ldots, \eta_r]$ the graded ideal corresponding to $\mathcal{N}$. Then $\mathrm{IndCoh}_{\mathcal{N}}(X)$ is the full subcategory of $\mathcal{F}$ such that all $\alpha \in I$ act locally nilpotently on $\mathcal{F}$, via the canonical map

$$k[\eta_1, \ldots, \eta_r] \to \bigoplus_n \underline{\mathrm{End}}(\mathcal{F}, \mathcal{F}[n]).$$

The assignment $\mathcal{N} \to \mathrm{IndCoh}_{\mathcal{N}}(X)$ is containment-preserving; in particular we have

$$\mathrm{QCoh}(X) = \mathrm{IndCoh}_{\emptyset}(X) \subseteq \mathrm{IndCoh}_{\mathcal{N}}(X) \subseteq \mathrm{IndCoh}_{\mathbf{P}^{n-1}}(X) = \mathrm{IndCoh}(X).$$

Our next goal is to generalize this to a setting without coordinates.

Remark 1.1. The smoothness of Y is essential. In the proof, we used this when we say that the pushforward of a coherent complex on X is perfect on Y.

2. Various constructions

2.1. **Hochschild cohomology.** Let $\mathcal{C}$ be a DG category. Let $Z(\mathcal{C}) = \underline{\mathrm{End}}_{\mathrm{End}(\mathcal{C})}(\mathrm{Id}_{\mathcal{C}})$, where $\mathrm{End}(\mathcal{C})$ is the monoidal DG category of DG functors $\mathcal{C} \to \mathcal{C}$.

Then $Z(\mathcal{C})$ is a DG algebra. Since $\mathrm{Id}_{\mathcal{C}}$ is the unit in $\mathrm{End}(\mathcal{C})$ for the composition, we get that $Z(\mathcal{C})$ is an algebra object in the category of algebras, i.e. an "E_2-algebra".

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Concretely, this means that $H^*(Z(\mathcal{C}))$ is a graded commutative algebra. This $Z(\mathcal{C})$ is called the *Hochschild cohomology* of $\mathcal{C}$.

How should we think about this? Suppose you have $\eta \in Z(\mathcal{C}) = \underline{\mathrm{End}}(\mathrm{Id}_{\mathcal{C}})$. Then η can be thought of as a collection of maps $\eta\colon \mathcal{F} \to \mathcal{F}$ for all $\mathcal{F} \in \mathcal{C}$, natural in $\mathcal{F}$.

Similarly, $\eta \in Z(\mathcal{C})[n]$ can be thought of as $\eta\colon \mathrm{Id}_{\mathcal{C}} \to \mathrm{Id}_{\mathcal{C}}[n]$ inside $\mathrm{End}(\mathcal{C})$, i.e. a collection of $\mathcal{F} \to \mathcal{F}[n]$ for all $\mathcal{F}$, natural in $\mathcal{F}$.

Example 2.1. If X is a suitably finite DG scheme, then we define $Z(X) := Z(\mathrm{QCoh}(X))$, and $Z(X) \cong Z(\mathrm{IndCoh}(X))$. The idea is that if you have a functor $\eta\colon \mathcal{F} \to \mathcal{F}[n]$ for all $\mathcal{F} \in \mathrm{QCoh}(X)$, then you get such a functor for each $\mathcal{C} \in \mathrm{Coh}(X)$, and then for all $\mathcal{F} \in \mathrm{IndCoh}(X)$.

2.2. The Hochschild-Kostant-Rosenberg map.

The HKR map goes

$$\Gamma(X, T_X[-1]) \to Z(X)$$

where $T_X \in \mathrm{QCoh}(X)$ is the tangent complex (dual to the cotangent complex).

Here is a construction of this map. Let X be a DG scheme. We form "Aut(X)" as some kind of group DG ind-scheme. Whatever this is, we should have an action of $\mathrm{Aut}(X)$ on $\mathrm{QCoh}(X)$.

Let G be a group DG ind-scheme. (It doesn't really matter that G is a group.) There is a construction $\Omega G = \mathrm{pt} \times_G \mathrm{pt}$, where the fiber product is taken in the derived sense. You could think of this as $\mathrm{Aut}_G(\mathrm{pt})$.

Then $\Omega\,\mathrm{Aut}(X)$ is "automorphisms of the identity automorphism of X", hence acts on $\mathrm{Id}_{\mathrm{Aut}(X)}$. By transport of structure, it then acts on $\mathrm{Id}_{\mathrm{QCoh}(X)}$.

Passing to Lie algebras, we get

$$\mathrm{Lie}(\Omega\,\mathrm{Aut}(X)) \to \underline{\mathrm{End}}_{\mathrm{End}(\mathrm{QCoh}(X))}(\mathrm{Id}_{\mathrm{QCoh}(X)}) = Z(X).$$

Here $\mathrm{Lie}(\Omega\,\mathrm{Aut}(X))$ is the tangent complex to $\Omega\,\mathrm{Aut}(X)$ at Id. Using the diagram

$$\begin{tikzcd}
\Omega\,\mathrm{Aut}(X) \arrow[r] \arrow[d] & \mathrm{pt} \arrow[d] \\
\mathrm{pt} \arrow[r] & \mathrm{Aut}(X)
\end{tikzcd}$$

we compute $\mathrm{Lie}(\Omega\,\mathrm{Aut}(X)) = T_{\mathrm{Aut}(X)}[-1] = \Gamma(X, T_X)[-1]$.

2.3. The hypersurface case.

Let $X = \{f = 0\} \subset Y$. In other words, we have a fiber square

$$\begin{tikzcd}
X \arrow[r, "i"] \arrow[d, "q"] & Y \arrow[d] \\
0 \arrow[r] & \mathbf{A}^1
\end{tikzcd}$$

We have a map

$$T_{X/Y} \to T_X \to i^* T_Y \xrightarrow{+1}$$

and

$$T_{X/Y} = q^* T_{0/\mathbf{A}^1} = \mathcal{O}_X[-1].$$

Hence we get a map $\mathcal{O}_X[-1] \to T_X$, which we can think of alternatively as $\xi \in \Gamma(X, T_X[1])$.

By the preceding discussion, we have $\Gamma(X, T_X[-1]) \to Z(X)$ and ξ is a class in degree 2, i.e. a point of $\Gamma(X, T_X[-1])[2]$, and $\xi \mapsto \eta \in Z(X)[2]$.

3. Singular support

We will now give a "coordinate-free" approach.

Given a DG scheme X, we can make a reduced (hence classical by definition) scheme $\operatorname{Sing}(X)$.

If X is affine, we define

$$\operatorname{Sing}(X) := \operatorname{Spec}\left(\bigoplus_n H^{2n}(Z(X))\right)^{\text{red}}.$$

We then define

$$\mathbf{P}\operatorname{Sing}(X) := \operatorname{Proj}\left(\bigoplus_n H^{2n}(Z(X))\right)^{\text{red}}.$$

We're going to give a key example where this is computable.

Definition 3.1. Let X be a finite type DG scheme. We say that X is *lci* (or *quasismooth*) if Ω^1_X (the cotangent complex) is Zariski-locally of the form $\operatorname{Cone}(P_1 \to P_2)$ where $P_i \in \operatorname{QCoh}(X)$ are projective, meaning locally direct summands of $\mathcal{O}_X^{\oplus n}$ (note that no shifts are allowed!).

Example 3.2. If X is smooth, meaning Ω^1_X is projective, then X is lci.

Example 3.3. If X arises as a fibered product of the form

$$\begin{tikzcd} X \arrow[r] \arrow[d] & Y \arrow[d] \\ W \arrow[r] & Z \end{tikzcd}$$

and Y, Z, W are smooth then X is lci (a simple calculation of the cotangent complex).

Fact 3.4. If X is lci, then étale locally $X \cong 0 \times_{\mathbf{A}^n} \mathbf{A}^m$ for some $f \colon \mathbf{A}^m \to \mathbf{A}^n$.

Remark 3.5. For any classical affine scheme X, there is an lci derived scheme $\widetilde{X}$ with X as its underlying classical scheme, by choosing a presentation of X of the above form and taking the derived fibered product instead.

Fact 3.6. For X lci affine,

$$\operatorname{Sing}(X) = \operatorname{Spec}\left(\operatorname{Sym} H^1(X^{\text{cl}}, T_X|_{X^{\text{cl}}})\right)^{\text{red}}.$$

The cotangent complex goes towards negative degrees and the tangent complex goes towards positive degrees. So $H^1(X^{\text{cl}}, T_X|_{X^{\text{cl}}})$ is the highest cohomology group, and measures the failure of X to be smooth.

Remark 3.7. Think of $\mathrm{Sing}(X)$ as analogous to the cotangent bundle of a smooth variety.

Example 3.8. If $X = Y \times_V 0$, for V a finite-dimensional vector space and Y smooth with $f\colon Y \to V$, we have $\mathrm{Sing}(X) \subset X^{\mathrm{red}} \times V^*$.

For $\mathcal{N} \subset \mathbf{P}\mathrm{Sing}(X)$, we define a subcategory $\mathrm{IndCoh}_{\mathcal{N}}(X) \subset \mathrm{IndCoh}(X)$. Corresponding to $\mathcal{N}$ is a graded ideal $I \subset H^*(Z(X))$ and $\mathrm{IndCoh}_{\mathcal{N}}(X) \subset \mathrm{IndCoh}(X)$ is the subcategory where homogeneous elements of I act locally nilpotently.

Example 3.9. If $\mathcal{N} = \mathbf{P}\mathrm{Sing}(X)$ then $\mathrm{IndCoh}_{\mathcal{N}}(X) = \mathrm{IndCoh}(X)$.

Example 3.10. If $\mathcal{N} = \emptyset$ then $\mathrm{IndCoh}_{\mathcal{N}}(X) = \mathrm{QCoh}(X)$. To see this, one reduces to the global complete intersection case by étale descent, and then it follows from our earlier discussion.