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Name: Tony Feng **Email/Phone:** tonyfeng@stanford.edu

Speaker's Name: Sam Raskin

Talk Title: The notion of singular support in DAG and its applications I

Date: 2 / 4 / 19 **Time:** 4 : 00 am / pm (circle one)

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COHERENT SINGULAR SUPPORT

SAM RASKIN

1. Meta-overview of mathematical research

(1) The first step is to have something you want to be true.
(2) The second is to calculate something, and usually you realize that your initial dream has complications.
(3) The third step is to salvage what you can, which is where technical stuff happens.

The subject of coherent singular support is technical in nature, concentrated in step (3). But I want to start with (1).

2. What we want

Let k be a field of characteristic 0. We recall the dual numbers $k[\epsilon] := k[\epsilon]/\epsilon^2$.

Lemma 2.1. *There is an equivalence between flat $k[\epsilon]$-modules and extensions*

$$0 \to V \to \mathcal{E} \to V \to 0$$

where V is a vector space over k.

Proof. Given $0 \to V \to \mathcal{E} \to V \to 0$ you get a $k[\epsilon]$-module $\mathcal{E}$ where multiplication by ϵ is the composition $E \to V \hookrightarrow E$.

Conversely, if M is a flat $k[\epsilon]$-module we get an extension

$$0 \to V = M/\epsilon \xrightarrow{\epsilon} M \to V \to 0.$$

□

We can try to extend this by working in derived categories, and dropping the word "flat".

The hope is then to get an equivalence of the form

$$k[\epsilon] - \text{mod} \cong \{V \in \text{Vec}, \eta \colon V \to V[1]\}.$$

Here everything is occurring in some suitable derived category.

However this is wrong, and for somewhat subtle reasons. In order to explain why, we have to give a digression about how to do these sorts of calculations. (It's possible to make a mistake and think that you proved this equivalence.)

Date: February 4, 2019.

3. Yoga of derived categories

Let $\mathcal{C}$ be a dg category. To first approximation, this means a category enriched over chain complexes. But we really want to view $\mathcal{C}$ as an object in the infinity category of dg categories, which means that for all practical purposes we cannot distinguish quasi-isomorphisms from chain complexes. (By contrast, when we look at chain complexes, it makes sense to make this distinction.)

Assume that $\mathcal{C}$ has all (homotopy) colimits, (equivalently, all direct sums).

Definition 3.1. An object $\mathcal{F} \in \mathcal{C}$ is *compact* if $\mathrm{Hom}(\mathcal{F}, -): \mathcal{C} \to \mathrm{Vect}$ commutes with all colimits (equivalently, all direct sums). (Here Vect is the derived category of vector spaces, i.e. chain complexes.)

Example 3.2. Think of compactness as a "smallness" condition, analogous to "finitely presented".

Example 3.3. Let A be a ring. Then $A \in A - \mathrm{mod}$ is compact.

Definition 3.4. A category $\mathcal{C}$ is *compactly generated* if $\mathrm{Hom}_{\mathcal{C}}(\mathcal{G}, \mathcal{F}) = 0$ for all compact $\mathcal{G} \in \mathcal{C}$ implies that $\mathcal{F} = 0$.

In this case, if $\mathcal{C}^c \subset C$ is the subcategory of compact objects, then you can recover $\mathcal{C}$ canonically via the "ind-category" construction: $\mathcal{C} = \mathrm{Ind}(\mathcal{C}^c)$.

Example 3.5. $(A\mathrm{-mod})^c = \mathrm{Perf}(A)$, the smallest subcategory of A-mod containing A and closed under shifts, finite colimits, and direct summands.

Definition 3.6. $\mathcal{F} \in \mathcal{C}$ is a *compact generator* if $\mathcal{F} \in \mathcal{C}^c$ and $\underline{\mathrm{Hom}}_{\mathcal{C}}(\mathcal{F}, \mathcal{G}) = 0 \implies \mathcal{G} = 0$.

Example 3.7. $A \in A - \mathrm{mod}$ is a compact generator.

Conversely, if $\mathcal{F} \in \mathcal{C}$ is a compact generator then $\mathcal{C} \cong A - \mathrm{mod}$ where $A = \underline{\mathrm{End}}_{\mathcal{C}}(\mathcal{F})$, via the functor $\underline{\mathrm{Hom}}(\mathcal{F}, -)$.

4. The hope, revisited

We now reformulate our hope: let B the tensor algebra $T(k[-1])$, i.e. the free dg algebra on $k[-1]$. Then our hope is that

$$k[\epsilon] - \mathrm{mod} \cong B - \mathrm{mod}.$$

To prove this, it would be enough to find a compact generator inside $k[\epsilon] - \mathrm{mod}$ and then show that its endomorphisms are B.

Maybe the first thing to try is $k[\epsilon]$, but this will just give the tautological identification $k[\epsilon] - \mathrm{mod} \cong k[\epsilon] - \mathrm{mod}$.

The interesting thing to try is $\mathcal{F} = k$, where ϵ acts by 0. A quick calculation shows that $\underline{\mathrm{End}}_{k[\epsilon]}(k) = B$. To compute this, use your favorite resolution of k as a $k[\epsilon]$-module:

$$\ldots \to k[\epsilon] \xrightarrow{\epsilon} k[\epsilon] \xrightarrow{\epsilon} k[\epsilon] \to 0$$

The $\underline{\mathrm{End}}_{k[\epsilon]}(k)$ will be a B-module, so any element gives a map $B \to \underline{\mathrm{End}}_{k[\epsilon]}(k)$, and then you need to check that this induces an isomorphism on cohomology.

Exercise 4.1. Check that this corresponds to what we did in the beginning.

However there is a problem: $k[\epsilon]$ is not compact in $k[\epsilon]-\mathrm{mod}$.

Proof. Define $\mathcal{F}_n$ to be the naive truncation

$$\mathcal{F}_n = 0 \to k[\epsilon] \xrightarrow{\epsilon} \ldots \xrightarrow{\epsilon} k[\epsilon] \to 0$$

Then $k = \varinjlim_n \mathcal{F}_n$, but if k were compact it would be a summand of some $\mathcal{F}_n$, and this would contradict the computation of its self-Ext. In other words, any perfect complex has the property that Hom out of it vanishes in sufficiently high homological degree. □

5. Rescuing the hope

Now we have to do something technical.

We define $\mathrm{Coh}(k[\epsilon]) \subset k[\epsilon]-\mathrm{mod}$ to be the full subcategory of bounded complexes with finite-dimensional cohomology. This contains $\mathrm{Perf}(k[\epsilon])$ strictly, since for example it contains k.

We now claim that

$$\mathrm{Ind}(\mathrm{Coh}(k[\epsilon])) \cong B - \mathrm{mod}.$$

Proof. Apply the previous argument (using that $\mathrm{End}_{\mathrm{Coh}}(k) = \mathrm{End}_{k[\epsilon]-\mathrm{mod}}(k) = B$), using that k is compact in $\mathrm{IndCoh}(k[\epsilon])$ by fiat, and it generates. □

We have an embedding

$$\mathrm{Perf}(k[\epsilon]) \hookrightarrow \mathrm{Coh}(k[\epsilon])$$

which induces a fully faithful functor (on applying Ind)

$$k[\epsilon] - \mathrm{mod} \hookrightarrow \mathrm{IndCoh}(k[\epsilon]) \cong B - \mathrm{mod}.$$

Lemma 5.1. $k[\epsilon]-\mathrm{mod}$ *corresponds to the full subcategory* $(B-\mathrm{mod})_{loc.\ nilp.}$ *of* $\mathcal{G} \in B-\mathrm{mod}$ *where*

$$\varinjlim(\mathcal{G} \to \mathcal{G}[1] \to \mathcal{G}[2] \to \ldots) = 0.$$

Proof. The functor $k[\epsilon] \to k$ induces the 0 map $k \to k[1]$. Since $k[\epsilon]-\mathrm{mod}$ is closed under colimits, everything lies in this subcategory. □

Example 5.2. There are two versions of "k" in $\mathrm{IndCoh}(k[\epsilon])$. One is gotten from $k \in \mathrm{Coh}(k[\epsilon])$, an the other is the colimit of the $\mathcal{F}_n$. They are different!

Moral: perfect complexes correspond to some kind of "local nilpotency" condition.

Remark 5.3. There is an alternative path we could have taken. If you take the category of flat $k[\epsilon]$-modules as a dg category, it is equivalent to $\mathrm{IndCoh}(k[\epsilon])$.

Generalization of this example: let Y be smooth over k. Let $f\colon Y \to \mathbf{A}^1$. Consider $X := \{f = 0\} = Y \times_{\mathbf{A}^1} 0$. (This is a derived scheme if f is not flat.) Let $\mathcal{F} \in \mathrm{QCoh}(X)$. There exists a canonical map $\mathcal{F} \to \mathcal{F}[2]$ naturally in $\mathcal{F}$, constructed as follows.

Notation: if $g\colon S \to T$ is a map of (suitably finite) k-schemes, there is a pullback functor

$$g^*\colon \ \mathrm{QCoh}(T) \to \mathrm{QCoh}(S)$$

and

$$g_*\colon \ \mathrm{QCoh}(S) \to \mathrm{QCoh}(T).$$

We have a triangle

$$\mathcal{F}[1] \to i^* i_* \mathcal{F} \xrightarrow{\lambda} \mathcal{F}.$$

(From the Koszul complex one can at least see that this is consistent with the size of $i^* i_* \mathcal{F}$.) The usual yoga then gives a map $\mathcal{F} \to \mathcal{F}[2]$.

We define $\mathrm{Coh}(X) \subset \mathrm{QCoh}(X)$ to be the full subcategory of bounded complexes with (locally) finitely generated cohomology groups. If X is a dg scheme with $\mathcal{O}_X$ bounded below (not always satisfied but true in our situation), then $\mathrm{Perf}(X) \subset \mathrm{Coh}(X)$. This then induces an embedding $\mathrm{QCoh}(X) \subset \mathrm{IndCoh}(X)$.

For formal reasons, η extends to IndCoh. For formal reasons, η extends to IndCoh, giving $\mathcal{F} \to \mathcal{F}[2]$.

Proposition 5.4. *We can identify* $\mathrm{QCoh}(X) \subset \mathrm{IndCoh}(X)$ *as the full subcategory of* $\mathcal{F}$ *where* η *acts nilpotently, i.e.* $\{\mathcal{F} \mid \varinjlim \mathcal{F}[2n] = 0\}$.

(In the example, η can be thought of as the obstruction to extending $\mathcal{F}$ to a first-order deformation.)

Proof. Let's show that if η acts nilpotently, then $\mathcal{F}$ comes from $\mathrm{QCoh}(X)$.

Claim: for all $\mathcal{F} \in \mathrm{IndCoh}(X)$, $\ker \eta \in \mathrm{QCoh}(X)$.

Proof: it suffices to study $\mathcal{F} \in \mathrm{Coh}(X)$. In this case $\ker \eta = i^* i_* \mathcal{F}$, which is always in $\mathrm{Perf}(X)$. This is because $i_* \mathcal{F} \in \mathrm{Coh}(Y) = \mathrm{Perf}(Y)$ by Serre's theorem (since Y is smooth over k). By induction we get that $\ker(\eta^n) = 0$ for all $n > 0$. Then $\mathcal{F} \cong \ker(\eta^\infty\colon \mathcal{F} \to \varinjlim \mathcal{F}[2n]) \in \mathrm{QCoh}$. □