MSRI
Mathematical Sciences Research Institute

17 Gauss Way Berkeley, CA 94720-5070 p: 510.642.0143 f: 510.642.8609 www.msri.org

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Tony Feng **Email/Phone:** tonyfeng@stanford.edu

Speaker's Name: Claire Voisin

Talk Title: Stable Birational Invariants

Date: 2 / 7 / 19 **Time:** 10 : 30 [✓ am] / pm (circle one)

Please summarize the lecture in 5 or fewer sentences:____

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

- ☐ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- ☐ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
  - **<u>Computer Presentations</u>**: Obtain a copy of their presentation
  - **<u>Overhead</u>**: Obtain a copy or use the originals and scan them
  - **<u>Blackboard</u>**: Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
  - **<u>Handouts</u>**: Obtain copies of and scan all handouts
- ☐ For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack <u>with this completed sheet on top</u> and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- ☐ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)
- ☐ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

STABLE BIRATIONAL INVARIANTS

CLAIRE VOISIN

1. Decomposition of the diagonal under degenerations

There are (unirational) smooth projective varieties which don't have a Chow decomposition of the diagonal, e.g. the Artin-Mumford double solid $\widetilde{X}_{f_0}$ has the property that $H^3_B(\widetilde{X}_{f_0};\mathbf{Z})_{\mathrm{tors}} \neq 0$ hence $\widetilde{X}_{f_0}$ does not have a cohomological decomposition of the diagonal, and is therefore not stably rational.

We now examine the stability of these properties under degeneration.

Theorem 1.1 (Voisin '14)**.** *Let $\mathcal{X} \to B$ be a projective flat morphism with fiber dimension ≥ 2. Assume the generic fiber is smooth and has a Chow decomposition of the diagonal. If the fiber X_0 has ordinary double points then $\widetilde{X_0}$ (the desingularization) has a Chow decomposition of the diagonal.*

Theorem 1.2. *Under the same setup, assume that the generic fiber $\mathcal{X}_t$ has a cohomological decomposition of the diagonal. Assume further that $H^{2*}_B(\widetilde{\mathcal{X}_0};\mathbf{Z})$ is algebraic. Then $\widetilde{\mathcal{X}}_0$ (the desingularization) has a cohomological decomposition of the diagonal.*

Proof. After making a base change, we assume that there is a section (x_t).

For very general t, there exists $D_t \subset \mathcal{X}_t$ and $\mathcal{Z}_t$ supported on $D_t \times \mathcal{X}_t$ such that

$$\Delta_t = \mathcal{X}_t \times x_t + \mathcal{Z}_t \in \mathrm{CH}^n(\mathcal{X}_t \times \mathcal{X}_t). \tag{1.1}$$

Perhaps after another base change, this data can be put in a family $\mathcal{D} \subset \mathcal{X}$, and $\mathcal{Z}$ supported in $\mathcal{D} \times_B \mathcal{X}$, with the relation (1.1) satisfied for very general t. By the closedness of the locus where a cycle is rationally equivalent to 0, it is satisfied for all t.

We conclude that for all $t \in B$,

$$\Delta(\mathcal{X}_t) = \mathcal{X}_t \times x_t + \mathcal{Z}_t \in \mathrm{CH}^n(\mathcal{X}_t \times \mathcal{X}_t)$$

even at $t = 0$. Then specializing to $t = 0$, we have that $\mathcal{Z}_0$ is supported on $\mathcal{D}_0 \times \mathcal{X}_0$, and gives a decomposition of the diagonal there. Now, X_0 has a set of ordinary double points $x_1, \ldots, x_n$. Consider the desingularization $\widetilde{X}_0$ obtained by blowing up these double points. The exceptional divisors are smooth quadrics Q_i. We have $\widetilde{\mathcal{X}}_0 \setminus \bigcup Q_i \cong \mathcal{X}_0 - \{x_1, \ldots, x_n\} =: U$.

So

$$\Delta_{\widetilde{\mathcal{X}_0}}|_{U \times U} = (\widetilde{\mathcal{X}}_0 \times x_0 + \mathcal{Z}_0)|_{U \times U} = 0 \in \mathrm{CH}^n(U \times U).$$

Date: February 7, 2019.

We then get by the localization exact sequence

$$\Delta_{\widetilde{\mathcal{X}_0}} = \widetilde{\mathcal{X}}_0 \times x_0 + \widetilde{\mathcal{Z}}_0$$

where $\widetilde{\mathcal{Z}}_0$ is supported in $D_0 \times \widetilde{\mathcal{X}}_0$ modulo cycles supported in the complement of $U \times U$, i.e. $\bigcup Q_i \times \widetilde{\mathcal{X}_0} \bigcup \widetilde{\mathcal{X}_0} \times Q_i$. We can write this as

$$\Delta_{\widetilde{\mathcal{X}_0}} = \widetilde{\mathcal{X}}_0 \times x_0 + \widetilde{\mathcal{Z}}_0 + \Gamma_1 + \Gamma_2$$

with $\Gamma_1 \in \mathrm{CH}^n(\bigcup Q_i \times \widetilde{\mathcal{X}_0})$ and $\Gamma_2 \in \mathrm{CH}^n(\widetilde{\mathcal{X}_0} \times \bigcup Q_i)$.

Now the key point is that the Q_i are quadrics, so we understand their Chow groups and the Chow groups of their products with other stuff. In particular, $\mathrm{CH}^n(\widetilde{\mathcal{X}_0 \times Q_i})$ is generated by product cycles

$$\sum_j W_j \times W'_j, \quad W_j \in \mathrm{CH}^*(\widetilde{\mathcal{X}_0}), \mathrm{CH}^*(Q_i).$$

It is then easy to conclude the proof. ☐

In conclusion, what we really used about quadratic singularities is that they gave rise to quadrics, and the Chow group of a product with a quadric is simple. This is because a quadric is rational. In fact, it would be enough in the case of isolated singularities that the exceptional divisors admit a decomposition of the diagonal. Colliot-Thélène and Pirutka described the general condition by saying that the desingularization map is CH_0-universally trivial.

Corollary 1.3. *The desingularization of a very general quartic double solid with* ≤ 7 *nodes has no cohomological (or Chow) decomposition of the diagonal, hence it is not stably rational.*

Proof. Suppose the quartic double solid is $X_f := V(y^2 = f)$. By the general theory of nodal K3 surfaces, such an X_f specializes to the Artin-Mumford X_{f_0}. Furthermore $H^4_B(\widetilde{X}_{f_0}, \mathbb{Z})$ is algebraic, so we can apply Theorem 1.2. ☐

However, $\widetilde{X_f}$ has $H^3_B(\widetilde{X_f}; \mathbf{Z})_{\mathrm{tors}} = 0$, so we lose the Artin-Mumford obstruction, and also $Z^4(\widetilde{X_f}) = 0$ [Voisin]. Together with the following Lemma, we got the vanishing of all the unramified groups for $\widetilde{X}_f$.

Lemma 1.4. $H^i_{nr}(X; A) = 0$ *for* $i > \dim X$*.*

Proof. $H^i_{nr}(X; A) = H^0(X_{Zar}, \mathcal{H}^i(A))$ where $\mathcal{H}^i(A)$ is the sheaf associated to the presheaf $U \mapsto U^i_B(U; A)$, which vanishes on affine U if $i > \dim U$. ☐

2. Abel-Jacobi map for codimension 2 cycles algebraically equivalent to 0

Let $X/\mathbf{C}$ be smooth. If $H^{3,0}(X) = 0$, we have the *intermediate Jacobian*

$$J^3(X) = \frac{H^{1,2}(X)}{H^3_B(X; \mathbf{Z})}.$$

The Hodge decomposition implies that this is a abelian variety, although there is no canonical polarization in general.

We have the Abel-Jacobi map

$$\varphi_X \colon \mathrm{CH}^2(X)_{hom} \to J^3(X)$$

by writing $Z = \partial\beta$ for a 3-cycle β (possibly by the homological triviality of Z), and sending Z to

$$\int_B \in H^{n-1,n-2}(X)^* \cong H^{1,2}(X).$$

The β is defined up to adding T with $\partial T = 0$, so everything is defined modulo integrals over such cycles.

Theorem 2.1 (Bloch)**.** *If* $\mathrm{CH}_0(X) = \mathbf{Z}$*, then* $\varphi_X \colon \mathrm{CH}^2(X)_{hom} \cong J^3(X)$.

This is slightly weird because the right side is an algebraic variety while the left is not (at least, a priori). So what to impose on the map? We ask that it be a "regular homomorphism", meaning for all B smooth and algebraic, and all $\zeta \in \mathrm{CH}^2(B \times X)$, the map

$$\Phi_\zeta \colon B \to J^3(X)$$

sending $b \mapsto \Phi_X(\zeta_b)$ is algebraic on B.

Question: does there exist a universal codimension 2 cycle on $J^3(X) \times X$? By this we mean $\zeta_{univ} \in CH^2(J^3(X) \times X)$ such that $\Phi_{\zeta_{univ}} \colon J^3(X) \to J^3(X)$ is the identity.

Example 2.2. In the codimension 1 case, $J^1(X) = \mathrm{Pic}^0(X)$ and there is such a universal cycle, namely the Poincaré divisor.

Proposition 2.3. *If X has a cohomological decomposition of the diagonal, then X admits a universal codimension 2 cycle.*

Proof. We can write

$$[\Delta_X] = [X \times x] + (j, \mathrm{Id}_X)_*[\widetilde{Z}] \in H^{2n}_B(X \times X; \mathbf{Z})$$

for some $j \colon \widetilde{D} \to X$. Hence for all $\alpha \in H^3_B(X; \mathbf{Z})$ we get

$$\alpha = j_*(\widetilde{Z}^*\alpha)$$

which is compatible with the various Abel-Jacobi homomorphisms

$$\begin{tikzcd}
CH^2(X)_{hom} \arrow[r, "\widetilde{Z}^*"] \arrow[d, "\varphi_X"] & CH^1(\widetilde{D})_{hom} \arrow[r, "j_*"] \arrow[d, "\varphi_{\widetilde{D}}"] & CH^2(X)_{hom} \arrow[d, "\varphi_X"] \\
J^3(X) \arrow[r, "{[\widetilde{Z}]^*}"] & J^1(\widetilde{D}) \arrow[r, "j_*"] & J^3(X)
\end{tikzcd}$$

and $j_* \circ [\widetilde{Z}]^* = \mathrm{Id}_{J^3(X)}$.

We have a universal cycle $\widetilde{D}_{univ}$ on $J^1(\widetilde{D}) \times \widetilde{D}$. We then take

$$(\mathrm{Id}_{J(X)}, j)_*([\widetilde{Z}]^*, \mathrm{Id}_{\widetilde{D}})^*\widetilde{D}_{univ}$$

to be the desired cycle on $J^3(X)$.

$\square$

3. 3-FOLDS

Suppose X is a 3-fold with $h^{1,0} = h^{3,0} = 0$. Then $J^3(X)$ is a principally polarized abelian variety. The polarization comes from the pairing on $H^3_B(X; \mathbf{Z})$.

Clemens-Griffiths showed that if X is rational, then $(J^3(X), \theta) = \bigoplus J(C_i)$ where C_i are curves.

Proposition 3.1. *If X has a cohomological decomposition of the diagonal, then the minimal class $\theta^{g-1}/(g-1)! \in H_B^{2g-2}(J^3(X); \mathbf{Z})$ is algebraic on $J^3(X)$, where $2g = b_3(X)$.*

Theorem 3.2. *Suppose X is rationally connected 3-fold. Then X has a decomposition of the diagonal if and only if the following conditions are satisfied.*

- *(i)* $H^*_B(X; \mathbf{Z})$ *has no torsion.*
- *(ii)* X *has a universal codimension 2 cycle.*
- *(iii)* $\frac{\theta^{g-1}}{(g-1)!}$ *is algebraic on* $J^3(X)$.

Example 3.3. Let X be a very general designularized quartic double solid with 7 nodes. Then X has no universal codimension 2 cycle. Why? We know that it has no decomposition of the diagonal. It is torsion-free, and the Jacobian has dimension 3, so it is necessarily the Jacobian of a curve, so (iii) is satisfied. So it must be (ii) that is not satisfied.