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Talk Title: Extending holomorphic forms from the regular locus of a complex space to a resolution

Date: 2 / 7 / 19 **Time:** 4 : 00 am / [✓ pm] (circle one)

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EXTENDING HOLOMORPHIC FORMS FROM THE REGULAR LOCUS OF A COMPLEX SPACE TO A RESOLUTION

CHRISTIAN SCHNELL

1. Motivating problem

Theorem 1.1 (Greb-Kebekus-Kovacs-Peternell). *Let X be a normal algebraic variety (over $\mathbf{C}$) with KLT singularities. Let $r\colon \widetilde{X} \to X$ be a resolution. Then every algebraic p-form on X_{reg} extends to an algebraic p-form on $\widetilde{X}$.*

In fact they show this even locally, in the following sense. Let $j\colon X_{\mathrm{reg}} \hookrightarrow X$. Then

$$r_*\Omega^p_{\widetilde{X}} \hookrightarrow j_*\Omega^p_{X_{\mathrm{reg}}}$$

is an isomorphism for all p.

The philosophy of the proof comes from the MMP. You show that there is an extension with some kind of pole, and then show that actually there are no poles.

We will give a general sufficient and necessary criterion for forms to extend, and we'll see that it actually has nothing to do with the MMP.

2. An example

We will discuss the example of cones.

Let $X \subset \mathbf{A}^N$ be the normalization of the cone over a smooth $Y \subset \mathbf{P}^{N-1}$. There's a resolution $r\colon \widetilde{X} \to X$, such that the fiber over the cone point is Y. This is the total space of $\mathcal{O}(-1)$ over Y.

Let Y be smooth and projective, of dimension $n-1 \geq 1$. Suppose L is ample. Let $X = \operatorname{Spec} \bigoplus_{m=0}^{\infty} H^0(Y; L^m)$. We have a resolution $r\colon \widetilde{X} \to X$, where $\widetilde{X}$ is the total space of L^{-1} over Y, i.e.

$$\widetilde{X} := \operatorname{Spec}_Y \left(\bigoplus_{m=0}^{\infty} L^m \right).$$

What are the n-forms on $\widetilde{X}$? We have $\omega_{\widetilde{X}} = q^*(\omega_Y \otimes L)$ hence

$$H^0(\widetilde{X}, \omega_{\widetilde{X}}) = \bigoplus_{m=1}^{\infty} H^0(Y, \omega_Y \otimes L^m).$$

On the other hand,

$$H^0(X_{\mathrm{reg}}, \omega_{\mathrm{reg}}) = H^0(\widetilde{X} - Y, \omega_{\widetilde{X}}) = \bigoplus_{m \in \mathbf{Z}} H^0(\omega_Y \otimes L^m).$$

Date: February 7, 2019.

Conclusion: all n-forms extend if and only if $H^0(Y, \omega_Y \otimes L^m) = 0$ for all $m \leq 0$. One can write down similar conditions for p-forms to extend, $p < n$, but they turn out to be dominated by this extension condition for n-forms.

Compare to the KLT condition: X is KLT if and only if Y is Fano, an $L \sim_{\mathbf{Q}} a(-K_Y)$ for $a > 0$. This is much more restrictive.

3. More general setup

Let X be a complex space, reduced of constant dimension n.

Locally, what this means concretely is that we have an open ball $B \subset \mathbf{C}^{n+c}$ and $X \subset B$ is defined by some holomorphic equations. Let $r \colon \widetilde{X} \to X$ be a resolution, which (it turns out) we can assume to be projective.

Problem: which holomorphic p-forms on X_{reg} extend to $\widetilde{X}$? Equivalently, describe $r_* \Omega^p_{\widetilde{X}} \hookrightarrow j_* \Omega^p_{X_{\text{reg}}}$.

Remark 3.1. $r_* \Omega^p_{\widetilde{X}}$ is independent of the resolution.

Example 3.2. For $p = 0$, the question is about holomorphic functions. When do holomorphic functions extend from X_{reg} to $\widetilde{X}$? Answer: exactly when codimension $X_{\text{sing}} \geq 2$. ($r_* \mathcal{O}_{\widetilde{X}}$ = functions on normalization.)

Example 3.3. For $p = n$, if X has rational singularities (in particular $r_* \mathcal{O}_{\widetilde{X}} \cong \mathcal{O}_X$ and $R^i r_* \mathcal{O}_{\widetilde{X}} = 0$ for $i \geq 1$), then $r_* \omega_{\widetilde{X}} \cong \omega_X$ by duality, hence ω_X is reflexive. This says that all n-forms on X_{reg} extend to $\widetilde{X}$. Fact: this is equivalent to $\operatorname{codim}_X(\operatorname{supp}(R^i r_* \mathcal{O}_{\widetilde{X}})) \geq i + 2$ for all $i \geq 1$.

The condition of having rational singularities include normal and Cohen-Macaulay. Asking for n-forms to extend amounted to rational singularities minus the part about normal and Cohen-Macaulay.

Theorem 3.4 (Kebekus-S)**.** *Let $B \subset \mathbf{C}^{n+c}$ be a ball, with coordinates $z_1, z_2, \ldots, z_{n+c}$. Let $X \subset B$ be reduced with constant dimension n. Pick a resolution $r \colon \widetilde{X} \to X$.*

(a) (well-known) Let $\alpha \in H^0(X_{\text{reg}}, \Omega^n_{X_{\text{reg}}})$. This extends to $\widetilde{X}$ if and only if the (n,n)-form $\alpha \wedge \overline{\alpha}$ is locally integrable. (Compare: a holomorphic function around a normal crossings divisor extends if and only if it is square-integrable.)

(b) (new) $\alpha \in H^0(X_{\text{reg}}, \Omega^p_{X_{\text{reg}}})$ extends to $\widetilde{X}^{an}$ if and only if all $\alpha \wedge dz_{i_1} \wedge \ldots \wedge dz_{i_{n-p}}$ and $d\alpha \wedge dz_{i_1} \wedge \ldots \wedge dz_{i_{n-p-1}}$ extend to $\widetilde{X}$, for all indices $i_1, \ldots, i_{n-p}$.

Consequences:

1. All n-forms extend implies all p-forms extend for all p.
2. All p-forms extend implies all $(p-1)$-forms extend.

Proof. Suppose α is a $(p-1)$-form. Then $d\alpha$ and $\alpha \wedge dz_i$ extend by assumption. Then by (b), $\alpha \wedge dz_{i_1} \wedge \ldots \wedge dz_{i_{n-p}}$ and $d\alpha \wedge dz_{i_1} \wedge \ldots \wedge dz_{i_{n-p-1}}$ extend. Using (b) again, α extends. ☐

4. Idea of proof

The proof uses Hodge modules and the Decomposition Theorem.

Why do this? Hodge theory is about holomorphic forms. For example, one part of the theory says that

$$H^0(X, \Omega^p_X) \subset H^p(X; \mathbf{C}).$$

One thing that Hodge modules do is give you this kind of result for morphisms.

Because we want to restrict our attention to non-singular spaces for the purposes of using D-modules, we call the composition

$$f \colon \widetilde{X} \xrightarrow{r} X \hookrightarrow B$$

and want to describe $f_*\Omega^p_{\widetilde{X}}$. The p-forms $\Omega^p_{\widetilde{X}}$ can be viewed as the result of taking the naive filtration on the holomorphic de Rham complex

$$\mathcal{O}_{\widetilde{X}} \xrightarrow{d} \Omega^1_{\widetilde{X}} \to \ldots \to \Omega^p_{\widetilde{X}} \to \ldots \to \Omega^n_{\widetilde{X}}.$$

Hodge modules on a complex manifold Y are "polarized variations of Hodge structure with singularities". The way this is implemented is: consider tuples (M, F_*M, K) where

- M is a regular holonomic $\mathcal{D}_Y$-module (perverse sheaf)
- F_*M is a filtration by coherent subsheaves, compatible with differentiation (including Griffiths transversality).
- K is constructible with $\mathbf{Q}$-coefficients, and $K \otimes_{\mathbf{Q}} \mathbf{C} \cong \mathrm{DR}(M)$.

Saito imposes a bunch of conditions, and shows that they come from VHS with singularities in the sense that they are extended over singularities from honest VHS on a smooth locus, and then pushed forward to the ambient space.

Example 4.1. The constant sheaf as a Hodge Module on Y is $(\mathcal{O}_Y, F_0\mathcal{O}_Y = \mathcal{O}_Y, \mathbf{Q}_Y[\dim Y])$.

Example 4.2. Let $X \subset Y$ be singular. The "intersection complex" as a Hodge Module on Y is

$$(M_X, F_*M_X, IC_X).$$

This is not so easy to describe. In practice the way it appears is that you consider a resolution, push down something simple from there, and take a "main" summand.

Notation: let $d = \dim Y$. The *de Rham* complex of a mixed Hodge module (M, F_*M, K) is

$$DR(M) = M \to \Omega^1_Y \otimes M \to \ldots \to \Omega^d_Y \otimes M$$

where the start is in degree $-d$.

This has a filtration

$$F_k DR(M) = F_k M \to \Omega^1_Y \otimes F_{k+1}M \to \ldots \to F_{k+d}M \otimes \Omega^d_Y$$

These are just $\mathbf{C}$-linear, but the graded pieces are $\mathcal{O}_Y$-linear

$$\mathrm{gr}^F_k DR(M) = \mathrm{gr}^F_k M \to \Omega^1_Y \otimes \mathrm{gr}^F_{k+1} M \to \ldots \to \Omega^d_Y \otimes \mathrm{gr}^F_{k+d} M$$

Theorem 4.3 (BBDG, Saito). *The pushforward $Rf_*\mathcal{O}_{\widetilde{X}}[n]$ breaks into simple pieces:*

$$Rf_*\mathcal{O}_{\widetilde{X}}[n] \cong IC_X \oplus \textit{(other terms supported on } X_{sing}\textit{)}.$$

This implies

$$Rf_*DR(\mathcal{O}_{\widetilde{X}}) \cong DR(M_X) \oplus (\text{other terms supported on } X_{sing}).$$

We're interested in differential forms, which have to do with the filtration. Crucially, Saito's version is compatible with the filtration and associated graded.

$$Rf_*\Omega^p_{\widetilde{X}}[n-p] \cong Rf_*\mathrm{gr}^F_{-p}DR(\mathcal{O}_{\widetilde{X}}) \cong \mathrm{gr}^F_{-p}DR(M_X) \oplus \underbrace{R_{-p}}_{\text{supported on } X_{sing}}.$$

Since we just want p-forms, we take cohomology sheaves in dimension $n-p$. We get

$$f_*\Omega^p_{\widetilde{X}} \cong \mathcal{H}^{p-n}(\mathrm{gr}^F_{-p}DR(M_X)) \oplus \mathcal{H}^{p-n}(R_{-p})$$

and since the left side is torsion-free on its support, the term $\mathcal{H}^{p-n}(R_{-p})$ (which is torsion on this support) must be 0.

One can then translate this statement into the result that everything is governed by n-forms. This takes a bit of work, but is basically just bookkeeping.