Moduli spaces of algebraic varieties of general type

János Kollár

Princeton University

May, 2019

Short recap of history — in an innovative format

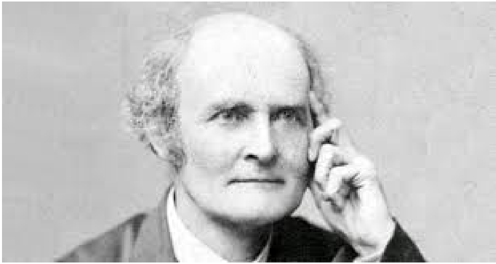
Started 160 years ago

Long before the talkies appeared.

Silent lecture

Riemann (1857)
Theorie der Abel'schen Funktionen

- Riemann surfaces as branched covers of $\mathbb{CP}^1$,
- genus g surfaces depend on $3g - 3$ parameters,
- $H^0(C, L) \geq \deg L + 1 - g$.



Cayley (1862)

A new analytic representation of curves in space

- $C \subset \mathbb{P}^3 \mapsto \mathrm{Cayley}(C) \subset \mathrm{Grass}(\mathbb{P}^1, \mathbb{P}^3)$
  $\mathrm{Cayley}(C) = \{\text{lines } L : L \cap C \neq \emptyset\}$.
- (Moduli of space curves) $\hookrightarrow$ (divisors on Grassmannian).
- Now Cayley form is called Chow form.

Hurwitz (1891)
Über Riemann'sche Flächen mit gegebenen Verzweigungspunkten

- Hurwitz space: branched covers of $\mathbb{CP}^1$,
- M_g is irreducible over $\mathbb{C}$.
- Char p: by Deligne-Mumford (1969)

Klein (1897-1912 with Fricke)

Vorlesungen über die Theorie der automorphen Funktionen

- Riemann surfaces as quotients of the unit disc in $\mathbb{C}$,
- study of discrete subgroups of $\mathrm{PSL}_2(\mathbb{R})$.
- M_g exists as a real orbifold.

Severi (1915)
Sulla classificazione delle curve algebriche e sul teorema d'esistenza di Riemann

- return to algebraic theory: plane curves with nodes.
- M_g is unirational for $g \leq 10$.

Siegel (1935),
Über die analytische Theorie der quadratischen Formen

- A_g: analytic moduli of abelian varieties
- as quotient of the Siegel upper half plane.
- Reads like modern mathematics.

Teichmüller (1944)
Veränderliche Riemannsche Flächen

- Teichmüller space T_g: Riemann surfaces with marked generators of π_1.
- Treats both complex structure and moduli functor.

Music selected and performed by Aaron Bertram

- Verlinde conjecture,
- Quantum Schubert Calculus,
- $12 = 10 + 2 \times 1$ (with Abramovich)
- Tropical Nullstellensatz (with Easton)

Moduli objects

Curve case.

- Interior: smooth, projective, ample K.
- Boundary: nodal, projective, ample K.

Surface case.

- Interior: Du Val (=ADE), projective, ample K.
- Boundary: semi-log-canonical, projective, ample K.

Higher dimensional case.

- Interior: canonical singularity, projective, ample K.
- Boundary: semi-log-canonical, projective, ample K.

Stable curve/surface/variety.

Interior families — curves

$X \to S$ proper family of irreducible curves. Then

$s \mapsto \mathrm{Nor}(X_s) = \mathrm{Res}(X_s)$ form a smooth, proper family iff

$s \mapsto \mathrm{genus}\big(\mathrm{Res}(X_s)\big)$ is locally constant.

Interior families — higher dimensions

$\mathrm{CanRes}(X_s) :=$ canonical model of $\mathrm{Res}(X_s)$

Theorem

$X \to S$ proper family of irreducible varieties of general type. Assume that S is reduced, connected. Equivalent:

- *$s \mapsto \mathrm{CanRes}(X_s)$ form a flat, proper family.*
- *$s \mapsto H^0\big(\mathrm{Res}(X_s), \mathcal{O}(mK)\big)$ are all constant.*
- *$s \mapsto vol\big(\mathrm{Res}(X_s), K\big)$ is constant.*
- *$s \mapsto (K^n)$ is constant for $\mathrm{CanRes}(X_s)$.*

Interior families III.

Corollary (Siu, Kawamata, Nakayama)

Let $g : X \to S$ be flat, proper, fibers of general type, smooth (or with canonical singularities).
Then $s \mapsto \mathrm{Can}(X_s)$ form a flat, stable family.

Stable families I.

Curve case.

$X \to S$ flat, proper, fibers nodal with ample K.

Higher dimensional case.

$X \to S$ flat, proper, fibers slc with ample K.

NEED MORE!

Semi-log-canonical is not an open condition

Family of varieties in $\mathbb{P}^5_{\mathbf{x}} \times \mathbb{A}^2_{st}$:

$$X := \left(rank \begin{pmatrix} x_0 & x_1 & x_2 \\ x_1 + sx_4 & x_2 + tx_5 & x_3 \end{pmatrix} \leq 1 \right).$$

Claim: the following are equivalent:

- $K_{X_{st}}$ is $\mathbb{Q}$-Cartier
- $3K_{X_{st}}$ is Cartier
- either $(s,t) = (0,0)$ or $st \neq 0$.

Being stable is not even locally closed.

Stable families II.

Higher dimensional case.

$g : X \to S$ flat, proper, fibers slc with ample K AND

- If $S = DVR$: $K_{X/S}$ is $\mathbb{Q}$-Cartier.
- If S normal: $K_{X/S}$ is $\mathbb{Q}$-Cartier.
- If S reduced: equivalent to normalization (char 0).
- (KSB defn.) $\forall m > 0\ \exists L_m$ flat sheaf with S_2 fibers:

$$L_m \cong \omega_{X/S}^{\otimes m} \quad \text{on the Gorenstein locus of } g.$$

- (Viehweg defn.) $\exists m > 0$ and a line bundle L_m:

$$L_m \cong \omega_{X/S}^{\otimes m} \quad \text{on the Gorenstein locus of } g.$$

Stable families III.

Comparing V and KSB conditions:

- V version depends on m in char p.
- equivalent over reduced schemes in char 0 (not in char p).
- [K-Altmann, 2015] For cyclic quotients of surfaces
  - infinitesimal KSB-deformations all globalize,
  - there are many more infinitesimal V-deformations.

KSB-stability is representable

$f : X \to S$: flat family of normal varieties
of pure relative dimension,

Theorem

There is a monomorphism $i_S : S^{\mathrm{stable}} \to S$ *such that, for every* $g : T \to S$, *the following are equivalent*

1. *The pull-back* $f_T : X_T \to T$ *is KSB-stable.*
2. g *factors as*

$$g : T \to S^{\mathrm{stable}} \xrightarrow{i_S} S.$$

Moduli space of stable varieties

Theorem

The moduli functor of stable varieties has a coarse moduli space that is locally of finite type and satisfies the valuative criterion of properness.

Theorem (Karu, Alexeev, Hacon-McKernan-Xu)

The connected components are proper.

Theorem (Fujino, Kovács-Patakfalvi)

The connected components are projective.

Complete families: Semi-stable reduction

Curve case. (Kempf–Knudsen–Mumford–Saint-Donat)
$X \to S$ proper family of curves. There exist

- $S' \to S$ proper, generically finite and
- $X' \to S'$ birational to $X \times_S S'$,

such that $X' \to S'$ has reduced, nodal fibers.

Higher dimensional case.

(Abramovich, Karu, Temkin, Włodarczyk)

.... such that $X' \to S'$ has

reduced, normal crossing (almost) fibers.

Complete families of curves

Reduced, nodal curve determines the stable curve.

- Geometric: delete rtl tails and contract rtl bridges.
- Canonical ring: $C \mapsto \mathrm{Proj} \sum H^0(C, mK_C)$.
- Functorial: $g : X \to S$ flat, proper; reduced, nodal fibers,
  $\Rightarrow g^{\mathrm{stable}} : X^{\mathrm{stable}} \to S$.

Proof. Etale locally over $(0, S)$.
Take a divisor D that meets X_0 at all comps of $(X_0)^{\mathrm{stable}}$.

Claim: $R^1 g_* \mathcal{O}(mD) = 0$ for $m \gg 1$ so

$X^{\mathrm{stable}} = \mathrm{Proj}_S \sum_{m \gg 1} \mathcal{O}_X(mD)$.

Complete families of surfaces I.

Reduced, normal crossing surface does not determine a stable surface.

- $\sum H^0(S, mK_S)$ need not be finitely generated (K. 2011).

- A surface S (with quotient singularties) can have 2 deformations $X_i \to \mathbb{A}^1$ such that the central fibers of $X_i^{\text{stable}} \to \mathbb{A}^1$ are not isomorphic.

Corollary. Over a nodal curve $B = (xy = 0)$ there is $X \to B$ flat, reduced, quotient sings. fibers such that $X^{\text{stable}} \to B$ does not exist,
(not even after ramified base change).

Complete families of varieties II.

Theorem (K.-Nicaise-Xu)

$g : X \to S$ with reduced, slc fibers and normal generic fiber. If S is smooth then we get $g^{\text{stable}} : X^{\text{stable}} \to S$.
(+ commutes with dominant base changes)

- (Tsunoda, 1984) For smoothings of snc surfaces, we get a unique canonical model. (????)

Questions.

- KNX over normal bases?
- Only finitely many stable models?
- Tsunoda in higher dimension?

Moduli of pairs: objects

Stable pair: $(X, \Delta = \sum a_i D_i)$

- **Global condition:** $K_X + \Delta$ ample.
- **Local condition:** semi-log-canonical

  implies $0 \leq a_i \leq 1$ and $D_i \not\subset \mathrm{Sing}(X)$.

Canonical ring: $\sum_{m \geq 0} H^0\big(X, \mathcal{O}_X(mK_X + \lfloor m\Delta \rfloor)\big)$.

Moduli of pairs: families

Major problem:

In stable families $g : (X, \Delta) \to S$

$X \to S$ is flat but
$\Delta \to S$ is not flat.

Example: lines on families of quadric surfaces.

$Q := (x^2 - y^2 + z^2 - t^2w^2 = 0) \subset \mathbb{P}^3_{xyzw} \times \mathbb{A}^1_t$,

$L_t = (x - y = z - tw = 0)$ and $L'_t = (x + y = z - tw = 0)$.

Compute self-intersections:
$(aL_0 + bL'_0)^2 = \frac{1}{2}(a+b)^2$ and $(aL_g + bL'_g)^2 = 2ab$. So

- $(aL_0 + bL'_0)^2 \geq (aL_g + bL'_g)^2$,
- $aL_t + bL'_t$ Cartier on every fiber iff $a + b$ is even,
- $aL + bL'$ is globally Cartier iff equality holds.

Numerical Cartier condition; weak form

Theorem (K., Bhatt-de Jong)

– $f : X \to C$ is flat, projective,
– normal or S_2 fibers.
– D divisor such that each D_c is Cartier and ample. Then

1. *$c \mapsto (D_c^n)$ is upper semi-continuous and*
2. *D is Cartier iff the above function is constant.*

Numerical criterion of stability

Corollary

$f : (X, \Delta) \to S$ flat, projective, S reduced,
– fibers are semi-log-canonical with
– ample log-canonical class $K_{X_s} + \Delta_s$. Then

1. *$s \mapsto \left(K_{X_s} + \Delta_s\right)^n$ is upper semi-continuous and*
2. *f is stable iff $s \mapsto \left(K_{X_s} + \Delta_s\right)^n$ is locally constant.*

Not equivalent
$s \mapsto H^0\left(X_s, \mathcal{O}_{X_s}(mK_{X_s} + \lfloor m\Delta_s \rfloor)\right)$ is locally constant.

Coefficients $\geq \frac{1}{2}$ — I.

Principle. If (X, Δ) is semi-log-canonical and the coefficients of Δ are close to 1 then $\mathrm{Supp}\Delta$ is well behaved.

Theorem (K.-Kovács, 2010, 2018)

$\mathrm{Supp}\Delta^{=1}$ *is Du Bois.*

Theorem (K. 2014)

$\mathrm{Supp}\Delta^{>5/6}$ *is seminornal.*

Example: $\left(\mathbb{A}^2, \frac{5}{6}(x^2 = y^3)\right)$ is log-canonical.

Coefficients $\geq \frac{1}{2}$ — II.

Theorem (K. 2014)

$g : X \to S$ is stable then $\mathrm{Supp}\Delta^{>1/2}$ is flat over S.

Corollary

If all coefficients in Δ are $> \frac{1}{2}$ then the moduli of stable pairs (X, Δ) can be handled as

1. *flat families of varieties X plus*
2. *flat families of divisors on X.*

Coefficients $\geq \frac{1}{2}$ — III.

Theorem. [K. 2018] $g : X \to S$ is stable, S reduced and all coefficients in Δ are $\geq \frac{1}{2}$. Then:

1. The sheaves $\omega_{X/S}^{[m]}(\lfloor m\Delta \rfloor)$ are flat over S and commute with base change.
2. $s \mapsto \chi\big(X_s, \omega_{X_s}^{[m]}(\lfloor m\Delta_s \rfloor)\big)$ are locally constant.
3. If $\operatorname{coeff} \Delta \subset \{\frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots, 1\}$, then, $f_*\omega_{X/S}^{[m]}(\lfloor m\Delta \rfloor)$ is locally free and commutes with base change.

Caveat: Normal general fiber or relative dim. 2.

Main open question

What is the right moduli functor for general stable pairs (X, Δ)?

Known cases

- Reduced bases in char 0.
- Non-reduced bases: non-equivalent versions in char 0.
- Problems in char p, even over reduced curves.

Coefficients $\geq \frac{1}{2}$ — IV.

Localized version: Let $(X, H + \Delta)$ be lc pair, H is Cartier and $\operatorname{coeff} \Delta \subset [\frac{1}{2}, 1]$. Then $\omega_X^{[m]}(\lfloor m\Delta \rfloor)$ is S_3 along H.

History: Elkik, Fujino, Alexeev, Hacon

Method of proof:
– $g : Y \to X$ proper, $H \subset X$ Cartier, $H_Y := g^* H$.
– F a coherent sheaf on Y, S_3 along H_Y.
When is $g_* F$ S_3 along H?

Push-forward $0 \to F(-H_Y) \to F \to F|_{H_Y} \to 0$ to get

$$0 \to g_* F(-H) \to g_* F \to g_* \big(F|_{H_Y}\big) \to \mathcal{O}_X(-H) \otimes R^1 g_* F$$

Thus $g_* F$ is S_3 along H if (almost iff)

(a) $R^1 g_* F = 0$ and
(b) $g_* \big(F|_{H_Y}\big)$ is S_2 along H.

Coefficients $\geq \frac{1}{2}$ — V.

(a) $R^1 g_* F = 0$ and
(b) $g_*(F|_{H_Y})$ is S_2 along H.

Kodaira-type vanishing: (a) needs $F = K + \text{(positive)}$

(b) needs $F = \text{(negative)} + \text{(fractional)}$
Example: $g : S \to T$ birational map of normal surfaces, F exceptional. Then

F is g-negative $\Rightarrow$ F is effective $\Rightarrow$ $g_* \mathcal{O}_S(F) = \mathcal{O}_T$ is S_2.

Choosing $g : Y \to X$ small, the
fractional part gives some wiggle room.

Coefficients $\geq \frac{1}{2}$ — VI.

Question. Let (X, Δ) be an slc pair, $\operatorname{coeff} \Delta \subset [\frac{2}{3}, 1]$.
$x \in X$ codimension ≥ 3, not an lc center. Is

$$\operatorname{depth}_x \omega_X^{[m]}(\lfloor m\Delta \rfloor) \geq 3 \ ?$$