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NOTETAKER CHECKLIST FORM

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Name: Ori Katz **Email/Phone:** ORIKATZ.OK@gmail.com

Speaker's Name: Alex Haro

Talk Title: Effective bounds for the measure of rotations

Date: 10 / 11 / 18 **Time:** 11 : 00 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: Haro obtains an effective, almost optimal lower bound of the Lebesgue measure of the set of parameters that are conjugated to rigid rotation given a family of analytic circle diffeomorphisms. This measure is estimated using an a-posteriori KAM scheme that relies on quantitative conditions checkable via computer assistance. Thus, Haro shows that obtaining non-asymptotic lower bounds for applicability of KAM theorems is feasible in certain cases.

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Effective bounds for the measure of quasi-periodicity

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Hamiltonian systems, from topology to applications through analysis I

Berkeley, October 8-12, 2018

Characterization of dynamics in phase space

A fundamental question in Dynamical Systems is to identify properties of the solutions:

- linearization,
- stability,
- regular dynamics,
- chaos,
- diffusion.

An even more difficult (and relevant!) problem is to quantify the size of the regions in phase/parameter space where such properties are satisfied. Specially if the system is not in a perturbative regime.

Quasi-periodic motions in phase/parameter space

KAM theory concerns the existence and persistence of quasi-periodic motion (carried in smooth tori).

> *Given a particular system with non-perturbative parameters, and given a particular region of interest in phase/parameter space, what is the abundance of quasiperiodic solutions in that region?*

In spite of the vast literature and knowledge of KAM theory, the above question seems unreachable.

Our aim is to show that obtaining non-asymptotic lower bounds in KAM theory is feasible, provided one has an a-posteriori theorem to characterize the problem.

Measure of conjugacies of circle maps

Let $\alpha \in A \subset \mathbb{R} \to f_\alpha$ be a family of analytic circle diffeomorphisms. The map f_α is C^ω-conjugate to a rigid rotation if there exists a C^ω-diffeomorphism h such that

$$f_\alpha(h(x)) = h(x+\theta)\,, \qquad \theta = \rho(f_\alpha),$$

where $\rho(f_\alpha)$ is the rotation number of f_α.

Problem

Obtain (almost optimal) lower bounds for the measure of parameters $\alpha \in A$ such that the map f_α is C^ω-conjugate to a rigid rotation.

Notice that the answer of the problem is well-known in rotation space (Herman, Yoccoz). Here we are interested in parameter space! (For which Arnold had assymptotic results).

Why circle maps?

- This context contains all the fundamental difficulties of the problem. Thus, we do not need to consider standard (but cumbersome!) estimates regarding geometric constructions.
- For this problem, we can also obtain upper bounds computing resonant regions.
- This is the context where KAM theory is better characterized, and there are paradigmatic and well-studied examples. The problem was open even for these examples.

Our solution to the problem of measure estimates

We follow three steps:

- **Step I:** Obtain a quantitative a-posteriori theorem for the existence of a single conjugacy of angle θ.
- **Step II:** Obtain a quantitative a-posteriori theorem that control the dependence with respect to θ of the result in Step I.
- **Step III:** Reduce the hypotheses of the theorem in Step II to conditions that can be checked using a finite amount of computations.

As a corollary of these 3 steps, we obtain an effective lower bound of the measure, which holds after verifying a finite number of inequalities that depend on a finite input.

Step I: fixed rotation number

Assume that:

- $\alpha \in A \subset \mathbb{R} \mapsto f_\alpha$ is a family of C^ω-diffeomorphisms of the circle;
- h is a C^ω-diffeomorphism of the circle;
- θ is a Diophantine number, and r_θ is the corresponding rotation.

Theorem

*Under some mild and **explicit** conditions, if the error function*

$$e(x) = f_\alpha(h(x)) - h(x + \theta)$$

is small enough, then there exist a couple $(\bar{h}, \bar{\alpha})$ close to (h, α) such that $f_{\bar{\alpha}}$ is conjugate to r_θ, and the conjugacy is $\bar{h}$.

Moreover, we have explicit control in terms of the initial objects and Diophantine properties of θ.

- Notice that this is not a result “à la Herman” that holds for a general family (relaying in the characterization of the rotation number).
- Here we fix the family and consider the “dual problem”: for a given rotation number, we want to find (control) the map which is conjugate to such rotation.
- The proof of the a posteriori result consists in proving the convergence of a Newton scheme from the initial seed (h, α).

Step II: dependence on θ

Assume that:

- $\alpha \in A \subset \mathbb{R} \mapsto f_\alpha$ is a family of C^ω-diffeomorphisms of the circle.
- $\theta \in B \subset \mathbb{R} \mapsto h_\theta$ is a Lipschitz family of C^ω-diffeomorphisms of the circle.
- $\theta \in B \subset \mathbb{R} \mapsto \alpha(\theta) \in A$ be a Lipschitz function.

Theorem

*Under some mild and **explicit** conditions, if the family of errors*

$$\theta \in B \longmapsto e_\theta(x) = f_{\alpha(\theta)}(h_\theta(x)) - h_\theta(x+\theta)$$

is small enough, then there exists a Cantor set $\Theta \subset B$ and a Lipschitz function $\theta \in \Theta \mapsto \bar{\alpha}(\theta) \in A$ that labels C^ω-conjugate maps.

Moreover, the measure of $\bar{\alpha}(\Theta)$ is explicitly controlled in terms of the initial objects and Diophantine properties defining Θ.

Step II: dependence on θ

Some comments

- This is the involved part. Notice that the true family of conjugations $\theta \mapsto h_\theta$ is defined on a Cantor set, and it is smooth in the sense of Whitney.
- For the above reason, approximating h_θ requires a very sharp control on the interval of θ where the approximation is defined.
- For the purpose of estimating the measure, Lipschitz regularity is enough.
- In particular, we prove that $\bar{\alpha}$ is Lipschitz from below, and apply that

$$\mathrm{Leb}(\bar{\alpha}(\Theta)) \geq \mathrm{lip}_\Theta(\bar{\alpha})\,\mathrm{Leb}(\Theta).$$

Step III: finiteness of the hypotheses

As approximation of the family of conjugacies we will take $h_\theta(x)$ to be a polynomial of degree m in $\theta - \theta_0$ (for some θ_0) whose coefficients are trigonometric polynomials of order N in x.

$$h_\theta(x) = x + \sum_{s=0}^{m} h^{[s]}(x)(\theta - \theta_0)^s\,, \qquad \alpha(\theta) = \sum_{s=0}^{m} \alpha^{[s]}(\theta - \theta_0)^s\,.$$

As a general idea, we reduce all conditions to be checked to a finite number (of order $N \log(N)$) of arithmetic operations, keeping a sharp control on the corresponding estimates.

Step III: finiteness of the hypotheses

- Given an interval $B \subset \mathbb{R}$, and given constants $\gamma < 1/2$ and $\tau > 1$, compute a lower bound for the Lebesgue measure of $B \cap \mathcal{D}(\gamma, \tau)$.

- Given polynomial-like functions we need to obtain sharp bounds for the analytic norms of

$$\partial_x h_\theta \,, \qquad 1/\partial_x h_\theta \,.$$

  To do so, we use maximum modulus principles and FFT.
- We also need to compute sharp bounds of

$$1/\langle b_\theta \rangle \,,$$

  where $b_\theta(x) = \partial_\alpha f_{\alpha(\theta)}(h_\theta(x))/\partial_x h_\theta(x+\theta)$.

- We also need to obtain sharp bounds for Lipschitz constants in B of

$$h_\theta - \mathrm{id} \,, \qquad \partial_x h_\theta \,, \qquad \alpha(\theta)$$

Step III: finiteness of the hypotheses

- The most delicate part is to obtain sharp control of the anaylic norm and Lipschitz regularity of

  $$e_\theta(x) = f_{\alpha(\theta)}(h_\theta(x)) - h_\theta(x + \theta)\,.$$

  To do so, we use a combination of Faà di Bruno formulas and a explicit approximation theorem to control the discretization error in Fourier space.

  In many practical cases, in which the function f_α is elementary, one can resort in Automatic Differentiation formulas.

An example: Arnold's family

Consider the well-known family:

$$\alpha \in [0,1] \longmapsto f_{\alpha,\varepsilon}(x) = x + \alpha + \frac{\varepsilon}{2\pi}\sin(2\pi x)\,,$$

where $|\varepsilon| < 1$.

Denote the rotation number of this family as

$$\rho_\varepsilon : \alpha \in [0,1] \mapsto \rho(f_{\alpha,\varepsilon})$$

Defining $K_\varepsilon = [0,1]\backslash \mathrm{Int}(\rho_\varepsilon^{-1}(\mathbb{Q}))$:

- Arnold (1961) proved that $\mathrm{Leb}(K_\varepsilon) \to 1$ for $|\varepsilon| \to 0$.
- Herman (1979) proved that, for $0 < |\varepsilon| < 1$, K_ε is a Cantor set of positive measure.

But no quantitative estimates for this measure were known.

- Fix a value of ε so that we work in a non-perturbative setting, say $\varepsilon = 0.25$.
- We split the interval of rotations $[0, 1] = B_1 \cup B_2 \cup \ldots \cup B_n$.
- For each subinterval B_i, we compute truncated Lindstedt series $\theta \in B_i \mapsto (h_\theta(x), \alpha(\theta))$.
- For each subinterval B_i, we obtain Diophantine constants (γ, τ) such that

$$\mathrm{Leb}(\Theta_i) \geq 0.99 \cdot \mathrm{Leb}(B_i)\,, \qquad \Theta_i = \mathcal{D}(\gamma, \tau) \cap B_i$$

- We invoke the KAM theorem using $(B_i, h_\theta(x), \alpha(\theta), \gamma, \tau)$. If we succeed, we obtain a lower bound for $\mathrm{Leb}(\bar{\alpha}(\Theta_i))$. If we fail, we subdivide B_i (branch and bound procedure).

Theorem

We have the following lower and upper bounds for the measure of parameters that are conjugate to rotation:

$$0.860748 < \mathrm{Leb}(K_{0.25}) < 0.914161\,.$$

- The lower bound using the KAM estimates.
- The upper bound using rigorous enclosures of the p/q-resonances for $q \leq 20$.

The KAM estimates are able to capture at least 94.16% of the measure. Of course, most part of the underestimation (and computational bottleneck) corresponds to the resonances

$$0/1, \quad 1/2, \quad 1/3, \quad 2/3, \quad 1/4, \quad 3/4.$$

An example: Arnold's family

Measure estimates around the golden rotation

ε_0	N	$\mathrm{Leb}(\alpha_\infty(\Theta))/\mathrm{Leb}(B) >$
$1/2^7 = 0.0078125$	64	0.999533 $(\mathrm{Leb}(B) = 1/2^{12})$
$10/2^7 = 0.078125$	128	0.998661 $(\mathrm{Leb}(B) = 1/2^{12})$
$20/2^7 = 0.15625$	256	0.995996 $(\mathrm{Leb}(B) = 1/2^{14})$
$30/2^7 = 0.234375$	256	0.991461 $(\mathrm{Leb}(B) = 1/2^{14})$
$40/2^7 = 0.3125$	256	0.984921 $(\mathrm{Leb}(B) = 1/2^{15})$
$50/2^7 = 0.390625$	512	0.976080 $(\mathrm{Leb}(B) = 1/2^{17})$
$60/2^7 = 0.46875$	512	0.964547 $(\mathrm{Leb}(B) = 1/2^{17})$
$70/2^7 = 0.546875$	512	0.949686 $(\mathrm{Leb}(B) = 1/2^{18})$
$80/2^7 = 0.625$	1024	0.930482 $(\mathrm{Leb}(B) = 1/2^{19})$
$90/2^7 = 0.703125$	1024	0.905233 $(\mathrm{Leb}(B) = 1/2^{19})$
$100/2^7 = 0.78125$	1024	0.870752 $(\mathrm{Leb}(B) = 1/2^{20})$
$110/2^7 = 0.859375$	2048	0.819862 $(\mathrm{Leb}(B) = 1/2^{22})$
$120/2^7 = 0.9375$	4096	0.728697 $(\mathrm{Leb}(B) = 1/2^{25})$
$125/2^7 = 0.9765625$	16384	0.627992 $(\mathrm{Leb}(B) = 1/2^{28})$

Assymptotic measure estimates

Our KAM estimates can also be used in the perturbative regime, thus recovering Arnold's result:

Corollary

Consider a family of the form

$$f_\alpha(x) = x + \alpha + \varepsilon g(x)$$

Then for every $\varepsilon < \varepsilon_* = \mathcal{O}(\gamma^2)$ (explicitly computed) we have

$$\mathrm{Leb}(\alpha_\infty(\Theta)) \geq \left(1 - \varepsilon \frac{\mathfrak{C}\|g\|_\rho}{\gamma \rho^{\tau+1}}\right) \left(1 - 2\gamma \frac{\zeta(\tau)}{\zeta(\tau+1)}\right)$$

where $\mathfrak{C}$ is explicitly computed, $\Theta = [0,1] \cap \mathcal{D}(\gamma, \tau)$ and ζ is the Riemann zeta function.

Summary

Given a one-parameter family of analytic circle diffeomorphism, we obtain (almost optimal) lower bounds for the measure of parameters for which the corresponding diffeomorphism is analytically conjugate to a rigid rotation.

- This is done by reducing the problem to a finite number of computations using a finite input;
- The number of computations is feasible;
- The conditions are explicit and checkable in concrete problems;
- As a by-product, we obtain an enclosure of the rotation number.

Many thanks!