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NOTETAKER CHECKLIST FORM

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Name: Ian Coley **Email/Phone:** msri@iancoley.org

Speaker's Name: Sam Raskin

Talk Title: Affine Beilinson-Bernstein at the critical level for GL_2

Date: 3 / 26 / 19 **Time:** 9 : 30 **[✓ am] / pm (circle one)**

Please summarize the lecture in 5 or fewer sentences:
For a particular case of group, they prove affine Beilinson-Bertstein. First they set up how affine B-B comes from classical B-B and then develop the appropriate machinery to prove the main theorem

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AFFINE BEILINSON-BERNSTEIN AT THE CRITICAL LEVEL FOR GL_2

SAM RASKIN

The Outline:

1. Beilinson-Bernstein
2. Affine Beilinson-Bernstein
3. The Proof for GL_2
4. Relation to geometric Langlands

So we will explain the title word-by-word. Throughout, everything seen should be considered derived and over a field k of characteristic zero.

I. BB This was the last step in proving the following conjecture of Kazhdan-Lusztig:

$$\begin{tikzcd}
& \{\text{some combinatorics}\} \arrow[dl, dash, dashed, "\text{K-L}"'] \arrow[dr, dash, "\text{K-L after Deligne, Grothendieck}"] & \\
\{\text{representation theory}\} \arrow[rr, leftrightarrow, "\text{BB-localization}"] & & \{\text{geometry of the flag variety}\}
\end{tikzcd}$$

where dashed means conjectural. Concretely, let G be a reductive group (e.g. GL_n) and $B \subset G$ a Borel subgroup (e.g. upper triangular matrices), and $\mathfrak{g}$ the Lie algebra of G.

Theorem 1 (BB-localization)**.** Global sections gives an isomorphism between the category of D modules on the (smooth and projective) flag variety G/B and representations of $\mathfrak{g}$ with the same central character as the trivial $\mathfrak{g}$-representation. Denote all this $\Gamma\colon D(G/B) \to \mathfrak{g}\text{-mod}_0$.

Heuristic of the theorem: for a $\mathfrak{g}$-module M in $\mathfrak{g}\text{-mod}_0$, we have $M = \int_{b' \in G/B} M^{b'}$. So M may not have invariants with respect to B, but for the other Borel subgroups.

Notes by Ian Coley.

There was an early desire for an affine version of BB, and early results were a bit unsatisfactory.

finite dimensional setup	affine setup
G	$G(K)$, $K = k((t))$ the algebraic loop group
B	$G(\mathcal{O})$, $\mathcal{O} = k[[t]]$
G/B	$\mathrm{Gr}_G = G(K)/G(\mathcal{O})$
$\mathfrak{g}$	$\mathrm{Lie}(G(K)) = \mathfrak{g}((t)) := \mathfrak{g} \otimes_k k((t))$

Problem: if $\mathfrak{g}$ is semisimple, then $Z(U(\mathfrak{g}((t))) = k$, suitably interpreted, so the central character restriction doesn't make sense.

Correction: let $\kappa \colon \mathfrak{g} \otimes \mathfrak{g} \to k$ be an ad-invariant symmetric bilinear form. Then we obtain an extension

$$0 \to k \to \widehat{\mathfrak{g}}_\kappa \to \mathfrak{g}((t)) \to 0$$

which is split as vector spaces, with the bracket defined by the 2-cocycle $\mathfrak{g}((t)) \otimes \mathfrak{g}((t)) \to k$, $(\xi, \varphi) \mapsto \mathrm{Res}(\kappa(\xi, d\varphi))$.

The center of these new algebras was described by Feigin-Frenkel. Call crit the special κ given by $-1/2 \cdot$ Killing form.

Theorem 2 (F-F).
1. If $\mathfrak{g}$ is simple and $\kappa \neq \mathrm{crit}$, then $Z(U(\widehat{\mathfrak{g}}_\kappa)) = k$.
2. If $\kappa = \mathrm{crit}$, there exists a canonical isomorphism $\mathrm{Spf}(Z(U(\widehat{\mathfrak{g}}_{\mathrm{crit}}))) \to \mathrm{Op}_{\check{G}}$, where the righthand side is "opers" on the formal punctured disc $D^\circ = \mathrm{Spec}\, k((t))$.

These opers were defined by Drinfeld and Sokolov in the 80's and are $\check{G}$-bundles with connection on D° with extra structure.

Example 3. $\check{G} = \mathrm{SL}_2$, $\mathrm{Op}_{\check{G}}$ is (up to choices) the space of connections of the form

$$\left\{ d + \begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix} dt : f \in k((t)) \right\}$$

Remark 4. For general $\check{G}$, $\mathrm{Op}_{\check{G}}$ is noncanonically isomorphic to $\mathrm{Spec}(Z(U(\check{\mathfrak{g}})))(K)$ the loop group on that spectrum. This can be well enough understood and is a "ind-pro-affine space".

Now we have $\Gamma \colon D_{\mathrm{crit}}(\mathrm{Gr}_G) \to \widehat{\mathfrak{g}}_{\mathrm{crit}}$-mod. The central characters are "bounded" by the central character of the vacuum representation $\mathbb{V}_{\mathrm{crit}} := \Gamma(\mathrm{Gr}_G, \delta_1) = \mathrm{ind}_{k \oplus \mathfrak{g}[[t]]}^{\widehat{\mathfrak{g}}_{\mathrm{crit}}}(k)$.

Being nonderived for a moment, we have $Z(U(\widehat{\mathfrak{g}}_{\mathrm{crit}})) \cong \operatorname{Fun} \operatorname{Op}_{\check{G}}$, and if we consider the surjection of the lefthand side onto $H^0 \operatorname{End}(\mathbb{V}_{\mathrm{crit}})$, we obtain a category of *regular* opers, i.e. opers on the non-punctured disc. So we improve Γ to

$$\Gamma\colon D_{\mathrm{crit}}(\mathrm{Gr}_G) \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}$$

factoring through the quotient. But this can't yet be an equivalence because $\mathbb{V}_{\mathrm{crit}}$ has a large set of endomorphisms, but $(\mathrm{Gr}_G, \delta_1)$ doesn't have so many. So how do we account for this?

Theorem 5 (Beilinson-Drinfeld)**.** $\Gamma\colon D_{\mathrm{crit}}(\mathrm{Gr}_G) \to \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}$ is a morphism of $\operatorname{Rep} \check{G}$-module categories, where the structure comes from

- $\operatorname{Rep} \check{G} \to D_{\mathrm{crit}}(\mathrm{Gr}_G)^{G(\mathcal{O})}$ by geometric (monoidal) Satake, and this acts on $D_{\mathrm{crit}}(\mathrm{Gr}_G)$.
- $\operatorname{Rep} \check{G} \to \operatorname{QCoh}(\operatorname{Op}^{\mathrm{reg}}_{\check{G}})$ which acts on $\widehat{\mathfrak{g}}_{\mathrm{crit}}$-modules by pullback along $\operatorname{Op}^{\mathrm{reg}}_{\check{G}} \to \mathbb{B}\check{G}$.

Conjecture 6 (Frenkel-Gaitsgory)**.** If we enhance Γ to incorporate this action, then we have an equivalence

$$\Gamma^{\mathrm{enh}}\colon D_{\mathrm{crit}}(\mathrm{Gr}_G) \otimes_{\mathrm{Reg}_{\check{G}}} \operatorname{QCoh}(\operatorname{Op}^{\mathrm{reg}}_{\check{G}}) \xrightarrow{\sim} \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_{\mathrm{reg}}$$

Theorem 7 (Raskin)**.** The above conjecture holds for $G = \mathrm{GL}_2$.

Outline of the proof: uses the theory of loop group actions on (dg) categories. All 'categories' henceforth are dg categories. The following things are true for all groups G:

Γ^{enh} is a morphism of categories with $G(K)$-action. We know:

1. F-G showed that Γ^{enh} is always fully-faithful, and induces an equivalence on I°-equivariant objects, where $I^{\circ} = G(\mathcal{O}) \times_G N$, which sits inside of $I = G(\mathcal{O}) \times_G B$, which all sits inside $G(\mathcal{O})$. To prove F-G we need only to show essential surjectivity.
2. A folklore result polished up by R: Γ^{enh} is a equivalence on Whittaker categories. If $\mathcal{C}$ is a category with $G(K)$ action, we get a category
$$\operatorname{Whit}(\mathcal{C}) := \mathcal{C}^{N(K),\psi} \simeq \mathcal{C}_{N(K),\psi}$$
where $\psi\colon N(K) \to \mathbb{G}_a$ is a "suitably nondegenerate character". The equivalence between invariants and coinvariants was proved by Raskin.

Theorem 8 (R.)**.** The Whittaker category $\operatorname{Whit}(\widehat{\mathfrak{g}}_{\kappa}\text{-mod}_{\mathrm{reg}})$ is equivalent to $\mathcal{W}_{\kappa}$-mod.

F-F observed that $Z(U(\widehat{\mathfrak{g}}_{\mathrm{crit}}))$ is equivalent to $\mathcal{W}_{\mathrm{crit}}$, so we don't actually need to give a definition of what that thing is in the case that we care about. All in all, work of Frenkel-Gaitsgory-Vilonen and Mirković-Vilonen showed that

$$\mathrm{Whit}(\mathrm{Gr}_G) \xrightarrow{\sim} \mathrm{Rep}_{\check{G}}$$

is an equivalence, which implies that

$$\mathrm{Whit}(D_{\mathrm{crit}}(\mathrm{Gr}_G) \otimes_{\mathrm{Reg}_{\check{G}}} \mathrm{QCoh}(\mathrm{Op}_{\check{G}}^{\mathrm{reg}})) = \mathrm{QCoh}(\mathrm{Op}_{\check{g}}^{\mathrm{reg}})$$

On the other hand,

$$\mathrm{Whit}(\widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}) = \mathrm{QCoh}(\mathrm{Op}_{\check{G}})$$

so restricting to reg on both sides gives us again $\mathrm{QCoh}(\mathrm{Op}_{\check{g}}^{\mathrm{reg}})$. This implies that (in the conjecture) Γ^{enh} induces an equivalent on Whittaker categories (well, technically we've only shown they are abstractly equivalent but more work does show this).

How do we turn this into the result?

Theorem 9 (R.)**.** Let $G = \mathrm{PGL}_2$. If $\mathcal{C}$ is a (dg) category with a $G(K)$-action, define $\mathcal{C}_0 \subset \mathcal{C}$ to be the minimal dg-subcategory such that

- $\mathcal{C}_0$ is closed under colimits
- $\mathcal{C}_0$ is closed under the $G(K)$-action
- $\mathrm{Whit}(\mathcal{C}) \subset \mathcal{C}_0$
- $\mathcal{C}^{I^\circ} \subset \mathcal{C}_0$ (recall the definition of I° above)

Then $\mathcal{C}_0 = \mathcal{C}$.

Remark 10. Some remarks:

- The F-G Conjecture is a corollary by taking for $\mathcal{C}_0$ the essential image of Γ^{enh}.
- This is parallel to a classical result: if $\mathrm{PGL}_2(\mathbb{Q}_p)$ acts on V, an irreducible smooth representation, then V is 1-dimensional or $V_{N(\mathbb{Q}_p),\psi} \neq 0$.
- Good heuristics exist in geometric Langlands saying: the failure of $\mathrm{Whit}(\mathcal{C})$ to generate $\mathcal{C}$ under the $G(K)$-action is encoded (pretty precisely) in singularities of maps
$$\mathrm{LS}_{\check{P}}(D^\circ) \to \mathrm{LS}_{\check{G}}(D^\circ)$$
where P goes over the parabolic subgroups of G. For $G = \mathrm{PGL}_2$, $P = B$. Singularities in the above map come from $H^1_{\mathrm{dR}}(D^\circ; (\widecheck{\mathfrak{g}/\mathfrak{b}})_{P_{\check{B}}})$, where $P_{\check{B}} = (0 \to \sigma \to (\xi, \nabla) \to \sigma^\vee \to 0) \in \mathrm{LS}_{\check{B}}(D^\circ)$, and these groups are zero unless $\sigma^{\otimes 2} = \mathrm{id}$

- Finally, F-G predict that for $\sigma \in \mathrm{LS}_{\check{G}}(D^\circ)$, there exists some category $\mathcal{C}_\sigma$ on which $G(K)$ acts. Moreover, if we choose a $\chi \in \mathrm{Op}_{\check{G}}$ mapping to σ, then $\mathcal{C}_\sigma = \widehat{\mathfrak{g}}_{\mathrm{crit}}\text{-mod}_\chi$. In particular, the FG Conjecture is the case χ = trivial.