NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

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Speaker’s Name: Sam Raskin

Talk Title: Affine Beilinson-Bernstein at the critical level for GL_2

Date: 3 / 26 / 19 Time: 9 : 30 am (circle one)

Please summarize the lecture in 5 or fewer sentences:

For a particular case of group, they prove affine Beilinson-Bernstein. First they set up how affine B-B comes from classical B-B and then develop the appropriate machinery to prove the main theorem.

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AFFINE BEILINSON-BERNSTEIN AT THE CRITICAL LEVEL
FOR GL₂

SAM RASKIN

The Outline:

(1) Beilinson-Bernstein
(2) Affine Beilinson-Bernstein
(3) The Proof for GL₂
(4) Relation to geometric Langlands

So we will explain the title word-by-word. Throughout, everything seen should be considered derived and over a field $k$ of characteristic zero.

I. BB This was the last step in proving the following conjecture of Kazhdan-Lusztig:

\[
\{\text{some combinatorics}\} \quad \text{K-L} \quad \text{BB-localization} \quad \text{R-L after Deligne, Grothendieck} \quad \{\text{representation theory}\} \quad \{\text{geometry of the flag variety}\}
\]

where dashed means conjectural. Concretely, let $G$ be a reductive group (e.g. $GL_n$) and $B \subset G$ a Borel subgroup (e.g. upper triangular matrices), and $\mathfrak{g}$ the Lie algebra of $G$.

**Theorem 1** (BB-localization). Global sections gives an isomorphism between the category of $D$ modules on the (smooth and projective) flag variety $G/B$ and representations of $\mathfrak{g}$ with the same central character as the trivial $\mathfrak{g}$-representation. Denote all this $\Gamma: D(G/B) \to \mathfrak{g}\text{-mod}_0$.

Heuristic of the theorem: for a $\mathfrak{g}$-module $M$ in $\mathfrak{g}\text{-mod}_0$, we have $M = \int_{w \in G/B} M^w$. So $M$ may not have invariants with respect to $B$, but for the other Borel subgroups.

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Notes by Ian Coley.
There was an early desire for an affine version of BB, and early results were a bit unsatisfactory.

<table>
<thead>
<tr>
<th>finite dimensional setup</th>
<th>affine setup</th>
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<tbody>
<tr>
<td>$G$</td>
<td>$G(K), K = k((t))$ the algebraic loop group</td>
</tr>
<tr>
<td>$B$</td>
<td>$G(\mathcal{O}), \mathcal{O} = k[[t]]$</td>
</tr>
<tr>
<td>$G/B$</td>
<td>$\text{Gr}_G = G(K)/G(\mathcal{O})$</td>
</tr>
<tr>
<td>$\mathfrak{g}$</td>
<td>$\text{Lie}(G(K)) = \mathfrak{g}(t)) := \mathfrak{g} \otimes_k k((t))$</td>
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Problem: if $\mathfrak{g}$ is semisimple, then $Z(U(\mathfrak{g}((t)))) = k$, suitably interpreted, so the central character restriction doesn’t make sense.

Correction: let $\kappa: \mathfrak{g} \otimes \mathfrak{g} \to k$ be an ad-invariant symmetric bilinear form. Then we obtain an extension

$$0 \to k \to \widehat{\mathfrak{g}}_\kappa \to \mathfrak{g}((t)) \to 0$$

which is split as vector spaces, with the bracket defined by the 2-cocycle $\mathfrak{g}((t)) \otimes \mathfrak{g}((t)) \to k$, $(\xi, \varphi) \mapsto \text{Res}(\kappa(\xi, d\varphi))$.

The center of these new algebras was described by Feigin-Frenkel. Call crit the special $\kappa$ given by $-1/2 \cdot $ Killing form.

**Theorem 2** (F-F). (1) If $\mathfrak{g}$ is simple and $\kappa \neq \text{crit}$, then $Z(U(\widehat{\mathfrak{g}}_\kappa)) = k$.

(2) If $\kappa = \text{crit}$, there exists a canonical isomorphism $\text{Spf}(Z(U(\widehat{\mathfrak{g}}_{\text{crit}}))) \to \text{Op}_\widetilde{G}$, where the righthand side is “opers” on the formal punctured disc $D^\circ = \text{Spec} k((t))$.

These opers were defined by Drinfeld and Sokolov in the 80’s and are $\widetilde{G}$-bundles with connection on $D^\circ$ with extra structure.

**Example 3.** $\widetilde{G} = \text{SL}_2$, $\text{Op}_{\widetilde{G}}$ is (up to choices) the space of connections of the form

$$\left\{ d + \begin{pmatrix} 0 & f \\ 1 & 0 \end{pmatrix} dt : f \in k((t)) \right\}$$

**Remark 4.** For general $\widetilde{G}$, $\text{Op}_{\widetilde{G}}$ is noncanonically isomorphic to $\text{Spec}(Z(U(\widehat{\mathfrak{g}})))(K)$ the loop group on that spectrum. This can be well enough understood and is a “ind-pro-affine space”.

Now we have $\Gamma: D_{\text{crit}}(\text{Gr}_G) \to \widehat{\mathfrak{g}}_{\text{crit}}$-mod. The central characters are “bounded” by the central character of the vacuum representation $\mathbb{V}_{\text{crit}} := \Gamma(\text{Gr}_G, \delta_1) = \text{ind}_{\mathfrak{g} \otimes [t]}^{\mathfrak{g}}(k)$. 
Being nonderived for a moment, we have \( Z(U(\hat{\mathfrak{g}}_{\text{crit}})) \cong \text{Fun Op}_{\mathcal{G}} \), and if we consider the surjection of the lefthand side onto \( H^0 \text{End}(V_{\text{crit}}) \), we obtain a category of regular opers, i.e. opers on the non-punctured disc. So we improve \( \Gamma \) to

\[
\Gamma : D_{\text{crit}}(\text{Gr}_G) \to \hat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{reg}}
\]

factoring through the quotient. But this can’t yet be an equivalence because \( V_{\text{crit}} \) has a large set of endomorphisms, but \( (\text{Gr}_G, \delta_1) \) doesn’t have so many. So how do we account for this?

**Theorem 5** (Beilinson-Drinfeld). \( \Gamma : D_{\text{crit}}(\text{Gr}_G) \to \hat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{reg}} \) is a morphism of \( \text{Rep} \hat{G} \)-module categories, where the structure comes from

- \( \text{Rep} \hat{G} \to D_{\text{crit}}(\text{Gr}_G)^{G(\mathcal{O})} \) by geometric (monoidal) Satake, and this acts on \( D_{\text{crit}}(\text{Gr}_G) \).
- \( \text{Rep} \hat{G} \to \text{QCoh}(\text{Op}_G^{\text{reg}}) \) which acts on \( \hat{\mathfrak{g}}_{\text{crit}} \)-modules by pullback along \( \text{Op}_G^{\text{reg}} \to \mathbb{B} \hat{G} \).

**Conjecture 6** (Frenkel-Gaitsgory). If we enhance \( \Gamma \) to incorporate this action, then we have an equivalence

\[
\Gamma^{\text{enh}} : D_{\text{crit}}(\text{Gr}_G) \otimes_{\text{Reg(G)}} \text{QCoh}(\text{Op}_G^{\text{reg}}) \xrightarrow{\sim} \hat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{reg}}
\]

**Theorem 7** (Raskin). The above conjecture holds for \( G = \text{GL}_2 \).

Outline of the proof: uses the theory of loop group actions on (dg) categories. All ‘categories’ henceforth are dg categories. The following things are true for all groups \( G \):

\( \Gamma^{\text{enh}} \) is a morphism of categories with \( G(K) \)-action. We know:

1. F-G showed that \( \Gamma^{\text{enh}} \) is always fully-faithful, and induces an equivalence on \( I^o \)-equivariant objects, where \( I^o = G(\mathcal{O}) \times_G N \), which sits inside of \( I = G(\mathcal{O}) \times_G B \), which all sits inside \( G(\mathcal{O}) \). To prove F-G we need only to show essential surjectivity.
2. A folklore result polished up by R: \( \Gamma^{\text{enh}} \) is an equivalence on Whittaker categories. If \( \mathcal{C} \) is a category with \( G(K) \) action, we get a category

\[
\text{Whit}(\mathcal{C}) := \mathcal{C}^{N(K),\psi} \simeq \mathcal{C}_{N(K),\psi}
\]

where \( \psi : N(K) \to \mathbb{G}_a \) is a “suitably nondegenerate character”. The equivalence between invariants and coinvariants was proved by Raskin.

**Theorem 8** (R.). The Whittaker category \( \text{Whit}(\hat{\mathfrak{g}}_{\text{crit}}\text{-mod}_{\text{reg}}) \) is equivalent to \( \mathcal{W}_\kappa\text{-mod} \).
F-F observed that $Z(U(\mathfrak{g}_{\text{crit}}))$ is equivalent to $W_{\text{crit}}$, so we don’t actually need to give a definition of what that thing is in the case that we care about. All in all, work of Frenkel-Gaitsgory-Vilonen and Mirković-Vilonen showed that

$$\text{Whit}(\text{Gr}_G) \xrightarrow{\sim} \text{Rep}_G$$

is an equivalence, which implies that

$$\text{Whit}(D_{\text{crit}}(\text{Gr}_G) \otimes_{\text{Res}_G} \text{QCoh}(\text{Op}_{\text{reg}}^g)) = \text{QCoh}(\text{Op}_{\text{reg}}^g)$$

On the other hand,

$$\text{Whit}(\mathfrak{g}_{\text{crit}}\text{-mod}) = \text{QCoh}(\text{Op}_G)$$

so restricting to reg on both sides gives us again $\text{QCoh}(\text{Op}_{\text{reg}}^g)$. This implies that (in the conjecture) $\Gamma_{\text{enh}}$ induces an equivalent on Whittaker categories (well, technically we’ve only shown they are abstractly equivalent but more work does show this).

How do we turn this into the result?

**Theorem 9** (R.). Let $G = \text{PGL}_2$. If $\mathcal{C}$ is a (dg) category with a $G(K)$-action, define $\mathcal{C}_0 \subset \mathcal{C}$ to be the minimal dg-subcategory such that

- $\mathcal{C}_0$ is closed under colimits
- $\mathcal{C}_0$ is closed under the $G(K)$-action
- $\text{Whit}(\mathcal{C}) \subset \mathcal{C}_0$
- $\mathcal{C}^{I^c} \subset \mathcal{C}_0$ (recall the definition of $I^c$ above)

Then $\mathcal{C}_0 = \mathcal{C}$.

**Remark 10.** Some remarks:

- The F-G Conjecture is a corollary by taking for $\mathcal{C}_0$ the essential image of $\Gamma_{\text{enh}}$.
- This is parallel to a classical result: if $\text{PGL}_2(\mathbb{Q}_p)$ acts on $V$, an irreducible smooth representation, then $V$ is 1-dimensional or $V_{N(\mathbb{Q}_p),\psi} \neq 0$.
- Good heuristics exist in geometric Langlands saying: the failure of $\text{Whit}(\mathcal{C})$ to generate $\mathcal{C}$ under the $G(K)$-action is encoded (pretty precisely) in singularities of maps

$$\text{LS}_{\tilde{P}}(D^\circ) \to \text{LS}_{\tilde{G}}(D^\circ)$$

where $P$ goes over the parabolic subgroups of $G$. For $G = \text{PGL}_2$, $P = B$. Singularities in the above map come from $H^1_{\text{dR}}(D^\circ; (\tilde{\mathfrak{g}}/\mathfrak{b})_{P_\mathfrak{b}})$, where $P_\mathfrak{b} = (0 \to \sigma \to (\xi, \nabla) \to \sigma^\vee \to 0) \in \text{LS}_{\tilde{B}}(D^\circ)$, and these groups are zero unless $\sigma^{\otimes 2} = \text{id}$
Finally, F-G predict that for $\sigma \in \text{LS}_{\tilde{G}}(D^\circ)$, there exists some category $\mathcal{C}_\sigma$ on which $G(K)$ acts. Moreover, if we choose a $\chi \in \text{Op}_{\tilde{G}}$ mapping to $\sigma$, then $\mathcal{C}_\sigma = \hat{\Gamma}_{\text{crit-mod}}$. In particular, the FG Conjecture is the case $\chi = \text{trivial}$. 