NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

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Speaker’s Name: Akhil Mathews

Talk Title: p-adic algebraic K-theory and topological cyclic homology

Date: 3/27/19  Time: 9:30 am

Please summarize the lecture in 5 or fewer sentences:

Older and recent work has shown how TC is useful in computing algebraic K-theory. They show how to piece together I-adic and p-adic results into a general new invariant called Selmer K-theory.

CHECK LIST

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Joint with Clausen, based on work of Clausen-Mathew-Morrow and Clausen-
Mathew-Naumann-Noel.

I. Motivation for K-theory

Let $R$ be a commutative ring. One can associate a family of groups $K_i(R)$ such
that $K_0(R)$ is the Grothendieck group of finitely generated projective $R$-modules,
and the higher $K$-groups come as homotopy groups of some space $K(R)$

$K_0(R)$ is a literal group completion of isomorphism classes of finitely generated
projective $R$-modules, so one way to see the space $K(R)$ is as a homotopy-theoretic
group completion of the category of finitely generated projective $R$-modules.

Definition 1. (Definition/Construction) Given a stable $\infty$-category $\mathcal{C}$ (e.g. a dg
category), there’s a general construction of a spectrum $K(\mathcal{C})$. In particular, $\pi_0K(\mathcal{C})$
is the free abelian group on $x \in \mathcal{C}$ modulo cofibre sequences: if $x' \to x \to x''$ is
a cofibre sequence, this $[x] = [x'] + [x'']$. This has a precise universal property
[Blumberg-Gepner-Tabuada].

So given a quasicompact quasiseparated scheme $X$, define $K(X) := K(\text{Perf}(X))$.
If $X = \text{Spec } R$, it’s the same thing.

Theorem 2 (Thomason-Trobaugh). $X \mapsto X(K)$ is a sheaf for the Nisnevich topol-
ogy.

It’s difficult to compute in general, so most computations are done obliquely rather
than from the definition. New computations come usually from new (discovered)
properties. So: think of $X \mapsto K(X)$ like a cohomology theory on schemes, which
can be made more explicit via étale, prismatic, crystalline, etc.

Notes by Ian Coley.
But: algebraic K-theory depends only on $\text{Perf}(X)$, as opposed to the other cohomology theories – it’s a noncommutative invariant.

**Example 3.** If $f: Y \to X$ is smooth and proper, $\mathbb{R}f_*: \text{Perf}(Y) \to \text{Perf}(X)$ induces $f_*: K(Y) \to K(X)$. In general, this doesn’t come as freely (if at all).

**Example 4** (Thomason). Grothendieck’s absolute purity conjecture: a Gysin sequence in $\ell$-adic cohomology. The analogue in K-theory is dévissage.

**Example 5** (Voevodsky). If $X \subseteq \mathbb{C}$ is a variety, for each Zariski open $U \subset X$, consider $H^*(U, \mathbb{Z}/\ell)$. Then $\text{colim}_U H^*(U, \mathbb{Z}/\ell)$ is generated in degree 1. This is a special case of Bloch-Kato for $\mathbb{C}(X)$.

Focus: algebraic K-theory with torsion coefficients or profinitely completed $K^\wedge$.

The difficulty is that this is not a sheaf for the étale topology, so you can’t use Galois theory to compute $K(k)$ from its separate extensions.

The principle is that there is an approximation to K-theory that does satisfy étale descent and is easier to compute. We could just étale sheafify, but that loses some lovely features, like only depending on $\text{Perf}(X)$.

II. $\ell$-adic K-theory

**Theorem 6** (Suslin-Gabber). If $k$ is a separately closed field of characteristic $\neq \ell$, then $K(k)^\wedge \cong ku^\wedge_\ell$, noncanonically, and $\pi_2 K(k)^\wedge \cong \mathbb{Z}/(i)$. The same holds for rings $R$ which are strictly henselian with residue field $k$.

New principle: $\mathbb{A}^1$-homotopy invariance for smooth algebras.

From now on, everything is $\ell$-adically complete, $\ell > 2$.

**Construction 7** (Miller-Mahowald). There is a functor $L_{K(1)}: \text{Sp} \to \text{Sp}$. Explicitly, if $X \in \text{Sp}$, then $L_{K(1)}(X)/\ell = X/\ell[v_1^{-1}]$ where $v_1: \Sigma^{2t-2}(S^0/\ell) \to S^0/\ell$ is something purely stable homotopy theoretic.

**Theorem 8** (Thomason). Let $X$ be a scheme over $\mathbb{Z}[1/\ell]$ plus some finiteness. Then $L_{K(1)}K(-)$ satisfies étale descent. Moreover, there is a spectral sequence $H^s_{\text{ét}}(X, \mathbb{Z}/\ell(t)) \Rightarrow \pi_{2t-s} L_{K(1)}K(X)$.

**Example 9.** If $X/\mathbb{C}$ is a variety, $L_{K(1)}K(X) = KU^\wedge(X(\mathbb{C}))$.

By our principle, the last thing to ask is: is this a good approximation? Yes! Consider the map $\star: K(x)^\wedge \to L_{K(1)}K(X)$.
Theorem 10 (Rosenschon-Østvaer, after Voevodsky-Rost). Suppose $X$ has finite Krull dimension and for all $x \in X$, $\text{vcd}_R(k(x)) \leq d$. Then $\ast$ is an isomorphism on $\pi_\ast$ with $\ast \geq \max(d - 2, 0)$.

III. $p$-adic K-theory

Suslin-Gabber no longer applies – K-theory is not locally constant!

Example 11. New examples:

- $K((\mathbb{F}_p))_p^\wedge \simeq H\mathbb{Z}_p$, proven by Quillen, concentrated on degree zero.
- $K((\mathcal{O}_{\mathbb{C}_p}))_p^\wedge \simeq ku_p^\wedge$, proven by Niziol, so somehow this is the same as the $\ell$-adic case.

One complication is that we are no longer insensitive to nil-ideals. Our new friend is topological cyclic homology $TC$.

Construction 12 (Bökstedt-Hsiang-Madsen ’93). Given a stable $\infty$-category $\mathcal{C}$, one constructs a spectrum and a map $K(\mathcal{C}) \to TC(\mathcal{C})$. The definition of $TC$ is more complicated, but the computations are easier. It’s derived from THH, which is derived from regular $HH$.

Theorem 13 (Dundas-Goodwillie-McCarthy). If $(R, I)$ is a ring with a nilpotent ideal, then there exists a homotopy cartesian square

\[
\begin{array}{ccc}
K(R) & \longrightarrow & K(R/I) \\
\downarrow & & \downarrow \\
TC(R) & \longrightarrow & TC(R/I)
\end{array}
\]

Thus relative K-theory agrees with relative TC. This is also explored by Hesselholt-Madsen (etc) to do many computations.

Does this fit our principle? Let $R$ be a $p$-complete ring, $p$-adically, $K(R)_p^\wedge \to TC(R)_p^\wedge$ is a good approximation.

Theorem 14 (Geissar-Hesselholt, CMM). If $R$ is $p$-complete, $TC(R)_p^\wedge$ is étale $p$-adic K-theory in degrees $\geq 0$. Also, $K_p^\wedge \to TC_p^\wedge$ is an equivalence in large enough degrees (on rings that aren’t too large).

IV. Gluing things together
**Definition 15** (Clausen). For $\mathcal{C}$ a stable $\infty$-category, then define the Selmer K-theory $K^\text{Sel}(\mathcal{C}) := L_1 K(\mathcal{C}) \times_{L_1 TC(\mathcal{C})} TC(\mathcal{C})$, where $L_1$ is like $L_{K(1)}$.

**Example 16.** This glues up II and III nicely.

- $K^\text{Sel}(\mathcal{C})_\mathbb{Q} \simeq K(\mathcal{C})_\mathbb{Q}$
- $K^\text{Sel}(\mathcal{C})_\ell \overset{\sim}{\to} L_{K(1)} K(\mathcal{C})$ if $\ell$ is invertible on $\mathcal{C}$
- $K^\text{Sel}(R)_p^\wedge \overset{\sim}{\to} TC(R)_p^\wedge$ if $R$ is $p$-complete

So it looks great, but does it fit our principle? There exists a map $K \to K^\text{Sel}$ in general, but is it any good? Further, $K^\text{Sel}$ is purely categorical, but is it an étale sheaf?

**Theorem 17** (CM,CMNN). The construction $X \mapsto K^\text{Sel}(X)$ is an étale sheaf on spectral schemes.

**Theorem 18** (CM). $K^\text{ét} \to K^\text{Sel}$ is an equivalence in degrees $\geq -1$.

So: étale K-theory is (up to negative business) a noncommutative invariant.

**Definition 19.** Let $k$ be a field, $p$ a prime. Let $d_k$ be $\text{vcd}_p(k)$ if the characteristic of $k$ is not $p$, and let $d_k$ be $1 + \log_p[k : k^p]$ otherwise.

**Theorem 20** (CM). If $X$ is a quasicompact quasiseparated spectral scheme of finite Krull dimension and $d = \sup_{x \in X} d_{k(x)}$, then $K(X)_p^\wedge \to K^\text{Sel}(X)_p^\wedge$ is an isomorphism in degrees $\geq \max(d - 2, 0)$.

Ingredients: a technical issue called ‘hypercompleteness’ on étale sheaves of spectra; enhancement of Dundas-Goodwillie-McCarthy to spectral stuff.