MSRI LECTURES ON PSEUDODIFFERENTIAL OPERATORS

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Abstract. Rough notes for lectures at the MSRI introductory workshop in Fall 2019.

- Lecture 2 culminated in elliptic theory of two varieties:
  - Scattering, $\Psi^{m,l}$; prototype $\Delta + 1 \in \Psi^{2,0}$, weighted Sobolev spaces
  - Compact manifolds, Ψ^m; prototype $\Delta \in \Psi^2$, Sobolev spaces
- Fredholm between appropriate spaces $P : H^s \to H^{s-m}$
- Follows from estimates: $P \in \Psi^m(M)$ is elliptic if and only if $P^* \in \Psi^m(M)$ is elliptic

$$\|u\|_{H^s} \leq C(\|Pu\|_{H^{s-m}} + \|u\|_{H^{-N}}) \tag{1}$$

  where N is anything and

$$\|v\|_{H^s} \leq C(\|P^* v\|_{H^{s'-m}} + \|v\|_{H^{-N}}) \tag{2}$$

  where $s' = -(s-m)$
- **Principal symbol:** $\sigma_m(P)$
  - We'll assume $\sigma_m(P)$ has a homogeneous of degree m (with respect to dilations in ξ) representative, p a function on $T^*M \setminus 0$ where 0 is the zero section.
- ΨDOs can be used to localize in T^*M, phase space.
- Define $S^*M = (T^*M \setminus 0)/\mathbb{R}^+$; $(x, \xi, t) \mapsto (x, t\xi)$
- We care about behavior at infinity, so we compactify the fibers
- $T^*_z M \cong \mathbb{R}^n \hookrightarrow \overline{\mathbb{R}^n}$ by gluing in a sphere at ∞.
- $\xi = \rho^{-1}\omega$; $\rho = |\xi|^{-1}$, $\omega \in S^{n-1}$ and identify $\mathbb{R}^n - \{0\}$ with $(0,\infty)_\rho \times S^{n-1}$
  - Observe that $\rho \to 0$ is equivalent to $|\xi| \to \infty$
- Add in a sphere at $\rho = 0$; $\overline{\mathbb{R}^n} = \mathbb{R}^n \sqcup [0,\infty)_\rho \times S^{n-1}/\sim$ where $(\rho, \omega) \sim \rho^{-1}\omega$
- Then $T^*M \hookrightarrow \overline{T}^*M$, the fiber compactified version and $S^*M \cong \partial\overline{T}^*M$
- Given $A \in \Psi^0(M)$, think of $\sigma_0(A) = a$ as a function on $\partial\overline{T}^*M$.
- Given $\alpha \in \partial\overline{T}^*M$, A "microlocalizes" to a neighborhood of α.

Definition 0.1. *$\alpha \notin WF(u)$ if $\exists A \in \Psi^0(M)$ elliptic at α (i.e., $a = \sigma_0(A)$ is nonzero) s.t. $Au \in C^\infty$. We say $\alpha \notin WF^s(u)$ if $\exists A \in \Psi$ elliptic at α such that $Au \in H^s$.*

Proposition 0.2. *If $\alpha \notin WF(u)$, then $\exists U$ a neighborhood of α s.t. if $B \in \Psi^0$ with $WF'(B) \subset U$, then $Bu \in H^s$.*

- What's WF'? Answer in context of $\mathbb{R}^n$ and Kohn-Nirenberg version
- $\xi \neq 0$, $(x, \xi) \notin WF'(\mathrm{Op}_0(b))$ if $\exists$ open cone Γ such that $b \in S^{-\infty}$ in Γ (given by a Schwartz symbol)
- In compactified version, a cone is just a neighborhood.
- What if $P \in \Psi^m$ is not elliptic? That is, the principal symbol p vanishes somewhere; least nondegenerately ($p = 0$ at a point $\implies dp \neq 0$ at that point)
- Today: p is real; $\sigma_m(p - p^*) = p - p^* = 0 \implies p - p^* \in \Psi^{m-1}$ when operators are self-adjoint
- Basic phenomenon: propagation of estimates within $\Sigma = p^{-1}(\{0\})$.

Definition 0.3. *Bicharacteristics are integral curves of H_p, the Hamiltonian vector field, $H_p b = \{p, b\} = \sum_j (\frac{\partial p}{\partial \xi_j}\frac{\partial b}{\partial x_j} - \frac{\partial p}{\partial x_j}\frac{\partial b}{\partial \xi_j})$ inside Σ; this is an exciting place.*

- In general, H_p is homogeneous of degree $m - 1$
- Simple computation shows that H_p extends to a C^∞ vectorfield up to $\partial\overline{T}^* M$ which is tangent to $\partial\overline{T}^* M$.
- $WF(u)$ propagates along bicharacteristics

$$\|B_1 u\|_s \leq C(\|B_2 u\|_s + \|B_3 P u\|_{s-m+1} + \|u\|_{-N}) \tag{3}$$

Theorem 1. *(Hörmander) $WF^s(u) \setminus WF^{s-m+1}(Pu)$ is a union of maximally extended bicharacteristics in Σ. Outside of Σ, $\alpha \notin WF^{s-m}(Pu)$ implies $\alpha \notin WF^s(u)$.*

- Radial points at which H_p is a multiple of generators of dilations $\sum \xi_j \frac{\partial}{\partial \xi_j}$
- Compactified perspective: H_p vanishes means $\sum \xi_j \frac{\partial}{\partial \xi_j} = -\rho \frac{\partial}{\partial \rho}$
- Nicest situation: H_p flow has a source/sink structure
- At radial sources/sinks there are versions of propagation estimates
- There is a threshold quantity $s_0 (= \frac{m-1}{2}$ if $p - p^* \in \Psi^{m-2})$ s.t. $s > s_0$ implies (3) holds even without B_2 term and $s < s_0$ implies (3) holds with $WF'(B_2)$ in a punctured neighborhood of a radial set.
- Consequence: for such a flow starting at a region of high regularity, we propagate them to a punctured neighborhood of other radial sets and then into it: provided the inequality for s_0 is satisfied
- May need variable order/anisotropic Sobolev spaces
- An upshot of this is that $\|u\|_s \leq C(\|Pu\|_{s-m+1} + \|u\|_{-N}$ and similar estimate for the adjoint implies Fredholm theory applies
- Positive commutator estimates:

$$\langle Pu, Au \rangle - \langle Au, Pu \rangle = \langle (AP - P^*A)u, u \rangle = \langle ([A, P] + (P - P^*)A)u, u \rangle \tag{4}$$

which implies $\sigma_{m+m'-1}(i([A, P] + (P - P^*)A)) = -H_p a - 2\tilde{p}a$, where $H_p a$ has a definite sign