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NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Michael McNulty Email/Phone: mmcnulty@math.ucr.edu

Speaker's Name: Maciej Zworski

Talk Title: Scattering Theory lecture 2

Date: 9 / 4 / 19 Time: 3 : 30 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: This lecture began relating the trace of an Anosov flow to the corresponding Ruelle zeta function and the interpretation of that trace.

CHECK LIST

(This is **NOT** optional, we will **not pay** for **incomplete** forms)

☑ Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.

☑ Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.

- Computer Presentations: Obtain a copy of their presentation
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☑ When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list. (*YYYY.MM.DD.TIME.SpeakerLastName*)

☑ Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

Theorem (Selberg ... $\#12$ 2.2 12)

Suppose φ_t ... is an Anosov flow $V \to X$ ... $T_v V = E_s(v) \oplus E_u(v)$ ... $\|d\varphi_t v\| \le C e^{-\beta t}\|v\|$, $t > 0$ ... $v \in E_s(v)$ ...

and $T^{\#}_{\gamma}$ are lengths ... primitive closed orbits ... X ...

$$F(\lambda) = \prod_{\gamma}^{m+1} \left(1 - e^{i\lambda T^{\#}_{\gamma}}\right)$$ continues ... to $\mathbb{C}$

Dyatlov–Zworski ... 2016 ... $\exists L, t_e \; \forall \varepsilon > 0, T > t_e$

... $\{ \ldots \mid t_e \le t \le T, \; d(x, \varphi_t(x)) < \varepsilon \}$

$\le C \varepsilon^{n} e^{nLT}$

$n = \dim X$

$L : \|d\varphi_t|_{T_x X}\| \le C e^{L|t|}$ ...

... (2.30)

$N(T) = \# \{ \gamma : T^{\#}_{\gamma} \le T \} \le C e^{(2n-1)LT}$ $\quad N(T) \sim \frac{e^{hT}}{T}$

Step 1. $\zeta_R(\lambda)$ is ...

COMMENT: ... $(-1)^k$ ... $I - P_\gamma$ ... $d\varphi_t v = v \Rightarrow \det(I - P_\gamma) \ne 0$ ...

$$\zeta_R(\lambda) = \prod \left(1 - e^{i\lambda T^{\#}_{\gamma}}\right) = \exp\left(-\sum \sum_{m=1}^{\infty} \frac{1}{m} e^{i\lambda T^{\#}_{\gamma} m}\right) =$$

$$= \exp\left(-\sum_{\gamma} T^{\#}_{\gamma} e^{i\lambda T_{\gamma}} / T_{\gamma}\right) = \prod_{k=0}^{n-1} \exp\left(-\sum \frac{T^{\#}_{\gamma} e^{i\lambda T_{\gamma}} \operatorname{tr} \Lambda^k P_{\gamma}}{T_{\gamma} \det(I - P_{\gamma})}\right)$$

γ all closed orbits

2.2) then ... used

$$\det(I - P_\gamma) = \sum_{k=0}^{n-1} (-1)^k \operatorname{tr} \Lambda^k P_\gamma$$

$$P_\gamma = d\varphi_{-T_\gamma}\Big|_{E_s(x)\oplus E_u(x)} \quad x \in \gamma \qquad \boxed{\text{th. \& det. indep. of } x}$$

Poincaré map.

Orientability assumption $x \longmapsto E_s(x)$ is orientable

$$\Longrightarrow \det(I - P_\gamma) = (-1)^{\dim E_s} |\det(I - P_\gamma)|$$

(can be removed S. Shen)

$$\frac{d}{d\lambda}\log \zeta_R(\lambda) = \frac{1}{i}\sum_\gamma \frac{T_\gamma^\# e^{i\lambda T_\gamma} \operatorname{tr}\Lambda^k P_\gamma}{|\det(I - P_\gamma)|}$$

$$= \frac{1}{i}\int_0^\infty \sum_\gamma \frac{T_\gamma^\# \delta(t - T_\gamma) \operatorname{tr}\Lambda^k P_\gamma}{|\det(I - P_\gamma)|} \qquad u(t) \in \mathcal{D}'(\mathbb{R}_+)$$

maps '66 ... '75

Atiyah – Bott – Guillemin trace formula:

$$\mathcal{E}_0^k = \{ \cdot \in \mathcal{E}^k ;\ \iota_V \cdot = 0 \}$$

$$\Lambda^k T^* X$$

$$P : C^\infty(X, \mathcal{E}_0^k) \longrightarrow C^\infty(X, \mathcal{E}_0^k), \quad P = \frac{1}{i}\mathcal{L}_V$$

$$\text{"tr"}\, e^{-itP} = \sum_\gamma \frac{T_\gamma^\# \delta(t - T_\gamma) \operatorname{tr}\Lambda^k P_\gamma}{|\det(I - P_\gamma)|} \qquad \boxed{t > 0}$$

$$(\iota_V\, \mathcal{L}_V\, \cdot = \cdot, \quad d(\iota_V \cdot) + \iota_V\, d\cdot = \iota_V^2\, d\cdot = 0)$$

2.3 $\boxed{k=0}$ functions

$$\text{"}\operatorname{tr}_{\mathcal{H}}\text{" } e^{-itP} = \sum_{\sigma} \frac{T_\sigma^{\#}\,\delta(t-T_\sigma)}{|\det(I-P_\sigma)|}$$

"tr" φ_t^*

time dependent

$$\varphi_t^* f(x) = f(\varphi_t(x)) = \int_X K(t,x,y)\,f(y)\,dy$$

Schwartz ker

$$\text{"tr" } P_t^* = \int_X K(t,x,x)\,dx = \pi_* \iota_\Delta^* K$$

diagonal

$$\iota : (t,x) \longrightarrow (t,x,x) \qquad \pi : (t,x) \longrightarrow t$$
$$\mathbb{R}\times X \longrightarrow \mathbb{R}\times X\times X \qquad \mathbb{R}\times X \longrightarrow X$$

Example $\quad Au(x) = u(\alpha x)\quad , x\in\mathbb{R}$

$$Au = \int K_A(x,y)\,u(y)\,dy \qquad K_A(x,y) = \delta(\alpha x - y)$$

$y = \alpha x$

$$WF(K_A) = \{(x,y;-\alpha\eta,\eta)\}$$

$\iota^* K_A$ well def if $WF(K_A) \cap \{(x,x,\xi,-\xi)\} = \emptyset$

i.e. $\alpha \neq 1$ $\qquad N^*\Delta \qquad N^*\{y=\alpha x\} = \{(x,y,-\alpha\eta,\eta)\}$

$$\text{"tr" } K_A = \int \delta(\alpha x - x)\,dx = \frac{1}{|\alpha-1|}$$

2.4]

$$K = f_t^* K_{Id}$$

$WF(f^* u) \subset \{ (z, {}^t f'(z)\,\eta) : (f(z), \eta) \in WF(u) \}$

$$f : \mathbb{R} \times X \times X \to X \times X$$
$$(t, x, y) \to (\varphi_{-t}(x), y)$$

$$WF(K) \subset \{ (t, \varphi_t(y), y, -V(x)\cdot\eta, \xi, -{}^T d\varphi_t(y)\xi) : t \in \mathbb{R},\ y \in X,\ \xi \in T^*_{\varphi_t(y)} X \setminus 0 \}$$

$N^*(\{(t,x,x)\})$

$WF(K) \cap N^*\{(t,x,x)\} \ni \{(t, x, x, 0, \xi, -\eta)\}$

$\varphi_t(y) = y$, $-V(x)\eta = 0$,

$\xi = {}^T d\varphi_t \xi$, **But** $I - d\varphi_t|$ is invertible $t \neq 0$ $(t > 0)$

Hence $\operatorname{tr} \varphi_t^* = \int K_{\varphi_{-t}}(x, x)\, dx$ makes sense as an element $\mathcal{D}'(\mathbb{R}_+)$ (proof later)

Assume we have

$$\operatorname{tr}^b \varphi_t^* = \sum_\gamma \frac{T_\gamma^{\#}\, \delta(t - T_\gamma)\, \operatorname{tr} \wedge^k P_\gamma}{|\det(I - P_\gamma)|}$$

$+\ \mathcal{E}_0^k$ Statement

$$\zeta_K(\lambda) = \exp\left(- \sum_\gamma T_\gamma^{\#} \frac{e^{i\lambda T_\gamma}\, \operatorname{tr} \wedge^k P_\gamma}{T_\gamma\, |\det(I - P_\gamma)|} \right)$$

$$\zeta_R(\lambda) = \prod_k \zeta_k(\lambda)^{(-1)^k}$$

2.5)

$$\partial_\lambda \log \xi_0(\lambda) = \frac{1}{i}\int_0^\infty \mathrm{tr}^b\, f_{-t}^* \, e^{i\lambda t}\, dt$$

How to prove meromorphy of the LHS it is enough to prove that of the RHS with INTEGRAL RESIDUES

FORMAL ARGUMENT:

[illegible]

$$= (P - \lambda)^{-1}$$

Theorem (essentially Faure–Sjöstrand 2011)

$\exists\, G \in \mathcal{S}^{0+}(X)$, $\ldots$ s.t. for

$H_{sG} = e^{-sG} L^2$, $D_{sG} = \{u \in \mathcal{D}'(X) : u \in H_{sG}\ \&\ Pu \in H_{sG}\}$

$$P - \lambda : D_{sG} \longrightarrow H_{sG}, \quad \mathrm{Im}\,\lambda \geq -s/C$$

is a Fredholm operator of index 0. In particular

$$(P - \lambda)^{-1} : H_{sG} \to H_{sG} \quad \text{meromorphic in } \mathrm{Im}\,\lambda \geq -\frac{s}{C}$$

$$H^s \longrightarrow H^{-s}$$

2.6)

__Theorem__ (Lee 2011) Supp $\operatorname{Im}\lambda > -\delta$, near λ_0

$$(P-\lambda)^{-1} = R_{hol}(\lambda) + \sum_{j=1}^{J} \frac{(P-\lambda_0)^{j-1}\Pi}{(\lambda-\lambda_0)^j}$$

st. $R_{hol}(\lambda)$ is hol near λ_0, $\Pi : H_{sG} \to H_{sG}$ (comm. w P)

$WF'(R_{hol}(\lambda)) \subset \Delta(T^*X) \cup \Omega_+ \cup (E_u^* \times E_s^*)$

$WF'(\Pi) \subset E_u^* \times E_s^*$

__where__ $\Omega_+ = \{(e^{-tH_p}(x,\xi), x, \xi) : t \geq 0, p(x,\xi) = 0\}$

E_u^* dual to E_s sink for the lifted flow

E_s^* for E_u source for the lifted flow

__Proof of Guillemin's formula__ (will skip for now)

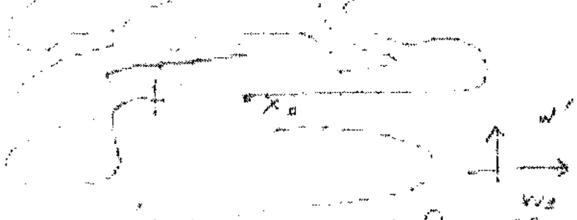
$$\int_{\mathbb{R}\times X} K(t,x,x)\chi(t,x) = \sum_\gamma \frac{1}{\det(I-P_\gamma)} \int_\gamma \chi(\gamma_t, x)\, d\ell(\gamma)$$

__Local version__ (hence dependence on x!)

$\varphi_{t_0}(x_0) = x_0$, $\quad \chi \in C_c^\infty((t_0-\epsilon, t_0+\epsilon) \times U)$

$\varphi_t(x_0) \in U$, $|t_0| < \epsilon$

$t_0 \ne 0$, $\quad A - P_\gamma \cdot$ ...



$\varphi_{-t_0}(0, w') = (F(w'), A(w'))$

$F(0) = 0$, $A(0) = 0$

$\varphi_{-t}(w_1, w') = (-t + t_0 + w_1 + F(w'), A(w'))$

$X = \partial_{w_1}$ on U

2.7) $K(t,x,x) = \delta(w' - A(w'))\,\delta(t - t_0 - F(w'))$

$x \in U$

$$\int_{\mathbb{R}\times Y} K(t,x,x)\,\chi(t,x)\,dx\,dt =$$

$$= \int \chi(t_0 - F(w'), (w_1, w'))\,\delta(w' - A(w'))\,dw_1\,dw'$$

$dA(0) = P_\gamma$ (conjugate to) $\quad A(w') = w'$ has one sol. $\boxed{A(0)=0}$

↑ differ

$$= \frac{1}{|I - dA(0)|} \int_{-\varepsilon}^{\varepsilon} \chi(t_0, w_1, 0)\,dw_1 =$$

$$= \frac{1}{|I - P_\gamma|} \int_\gamma \chi(T_\gamma, x)\,dL(x).$$

$dL(x) = dt$ measure on γ