MSRI LECTURES ON GEOMETRIC MICROLOCAL ANALYSIS
LECTURE 1

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ABSTRACT. Rough notes for lectures on geometric microlocal analysis at the MSRI introductory workshop in Fall 2019.

• Plan for this Lecture Series:
  (1) What is geometric microlocal analysis? Generalities plus a case study.
  (2) Further details about the case study
  (3) Further examples.
• Aphorisms:
  (1) When in doubt, compactify.
  (2) If you are still in doubt, blow something up.
  (3) Smoothness is not what it seems.
• We want to study PDEs on space which are:
  (1) Noncompact, complete manifolds with tame geometry at infinity like $\mathbb{R}^n$ and $\mathbb{H}^n$.
  (2) Singular spaces like cones, edges, stratified spaces, etc.
  (3) Spaces or operators which are degenerating; adiabatic limits, neck stretching, geometric gluing, etc.
• Examples of singular spaces: level sets of Morse functions; algebraic varieties; compactification of moduli spaces; compactification of locally symmetric spaces.
• Focus on elliptic operators: $L = \sum_{|\alpha| \leq m} a_\alpha(z) D_x^\alpha$. The principal symbol equals $\sigma_m(L)(z, \xi) = \sum_{|\alpha| = m} a_\alpha(z) \xi^\alpha$, and ellipticity means that this is invertible when $\xi \neq 0$.
• A priori estimates vs. parametrices:
  – The classical Sobolev estimates for an elliptic operator $\|u\|_{H^{s+m}} \leq (\|Lu\|_{H^s} + \|u\|_0)$ (strictly speaking, the norm on the left should be over a domain which is smaller than the one used on the right;
  – A parametrix for $L$ is an approximate inverse, namely a pseudodifferential operator $G \in \Psi^{-m}$ such that $LG = I - R_1$, $GL = I - R_2$ where the two operators $R_1, R_2 \in \Psi^{-\infty}$ are ‘residual’.

Existence of a pseudodifferential parametrix and knowledge of its mapping properties imply the Sobolev estimates:
* $Lu = f$ implies $u - R_1 u = Gf$, which gives $\|u\|_{H^{s+m}} \leq C(\|f\|_{H^s} + \|u\|_0)$ since $G : H^s \to H^{s+m}$ is bounded and $R_1 : H^t \to H^t$ is bounded for all $t, t'$

- One can use the a priori estimates plus a bit of functional analysis to deduce the existence of a (generalized) inverse for $L$, i.e., an inverse up to finite rank errors (the projects onto the kernel and cokernel). However, this does not tell you the structure of this generalized inverse.

- **Our goal:** a global theory of parametrices:
  - $L$ a differential operator on manifold $M$, $G$ a parametrix for $L$ with $G(z, z') \in D'(M \times M)$ its Schwartz kernel.
  - We care about the geometric structure of $G$
  - After compactifying, things become interesting at the boundary

- **Singular integral operators** $\frac{F(z)}{|z|^n}, \int_{S^{n-1}} F = 0$.
  - Oscillatory integral representation of $\Psi$DOs
  - Melrose and collaborators led to Schwartz kernels

- **Hadamard Parametrix Construction:** (due to Friedlander)
  - Given $L$, find the parametrix $G$, $G \sim \sum_{j=0}^{\infty} G_j$ with $LG_0 = I - R_0$
  - If $L = \sum_{|\alpha|=m} a_\alpha (z_0) D_\alpha^2$; $G_0 (z, \bar{z}) = \mathcal{F}^{-1} \left( \frac{1}{\sum_{|\alpha|=m} a_\alpha (z_0) \xi^\alpha} \right)$
  - For $\chi$ a suitable cutoff function, this is
    $$\int e^{i(z-\bar{z}) \cdot \xi} \frac{\chi(\xi)}{\sum_{|\alpha|=m} a_\alpha (z_0) \xi^\alpha} d\xi$$
  - For instance, if $m = 2$ and $L = \Delta$, then $\mathcal{F}^{-1} \left( \frac{1}{\xi^2} \right) = \frac{1}{|z - \bar{z}|^{n-2}}$
  - $LG_0 = I +$ error and $G_0 (z, \bar{z}) \sim a_0 (z) d(z, \bar{z})^{m-n}$
  - $LG_0 = I - R_0 (z, \bar{z})$, $R_0 (z, \bar{z}) \sim d(z, \bar{z})^{-m}$
  - $L(G_0 + \ldots + G_N) = I - R_N, R_N \sim d(z, \bar{z})^{-n+N+1} + C^\infty (M \times M)$.
  - Want an asymptotic sum: $G \sim \sum G_j, LG = I - \tilde{R}; \tilde{R} \in C^\infty (M \times M)$.
  - In this construction, any $M$ works but if $M$ is open or singular, then $\tilde{R}$ is not necessarily a compact operator.
  - $G^* L = I - \tilde{R}^t; \tilde{R}^t : \mathcal{E}'(M) \to C^\infty (M)$ does not improve growth.

- **Turn our attention to $\mathbb{H}^n$**:
  - Metric in half-space model:
    $$\frac{dx^2 + dy^2}{x^2}$$
  - Metric in Poincaré disk:
    $$\frac{4|dz|^2}{(1 - |z|^2)^2}$$
Metric in Klein model:
\[ \frac{4(\sum z_j dz_j^2)}{(1 - |z|^2)^2} + \frac{4|dz|^2}{1 - |z|^2} \]  
(3)

Laplacian in half-space:
\[ \Delta_y = x^2 \partial^2_x + (2 - n)x \partial_x + x^2 \Delta_y \]  
(4)
which degenerates at \( x = 0 \).

Laplacian in Poincaré:
\[ \Delta_g = (1 - |z|^2)^2 \Delta_x - 2(2 - n) \sum z_i \partial z_i \]  
(5)

\( (M, g) \) conformally compact with \( g = \rho^{-2} \bar{g}, \rho = 0 \) on \( \partial M, d\rho \neq 0 \).

\( (r, \theta) \) on \( \mathbb{H}^n \) with \( r \) distance from origin and \( \theta \in S^{n-1} \),
\[ g = dr^2 + \sinh^2(r) d\theta^2 \]  
(6)
and
\[ \Delta = \sinh^{1-n} r \partial_r (\sinh^{n-1} r \partial_r) + \frac{1}{\sinh^2 r} \Delta_\theta \]  
(7)

Want to solve
\[ G'' + (n - 1) \frac{\cosh r}{\sinh r} G' = 0, \quad r > 0 \]  
(8)

Thus, either \( G \sim r^0, r^{2-n} \).

With \( \rho = e^{-r} \), we have \( G \sim \rho^{n-1} \) as \( \rho \to 0 \), i.e., \( r \to \infty \).

At the boundary of \( M \times M \) compactified we have nice expansions with \( G \sim r^{2-n} \) toward the diagonal.

What happens at the corners of the diagonal?

 Blow up the corner along the diagonal

Dilation invariant: \( G(x, y, \tilde{x}, \tilde{y}) = G(\lambda x, \lambda y, \lambda \tilde{x}, \lambda \tilde{y}) \)

How to localize?

\( \mathbb{H}^n \setminus \Omega, \Delta + V, V \in C_0^\infty \), define \( \tilde{G} = \tilde{\chi}_1 G_{in} \chi_1 + \tilde{\chi}_2 G_{out} \chi_2 \) where \( \chi_1, \tilde{\chi}_1 \) compactly supported in \( U_1 \) and \( \chi_2, \tilde{\chi}_2 \) compactly supported in \( U_2 \) where \( \Omega \subset U_1 \subset U_2 \).

\( G_{out} = G_{\mathbb{H}^n}, G_{in} = \text{standard local } \Psi \text{DO}. \)

Want to obtain local parametrices and glue them together.