MSRI LECTURES ON GEOMETRIC MICROLOCAL ANALYSIS LECTURE 1

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ABSTRACT. Rough notes for lectures on geometric microlocal analysis at the MSRI introductory workshop in Fall 2019.

- Plan for this Lecture Series:
  (1) What is geometric microlocal analysis? Generalities plus a case study.
  (2) Further details about the case study
  (3) Further examples.
- Aphorisms:
  (1) When in doubt, compactify.
  (2) If you are still in doubt, blow something up.
  (3) Smoothness is not what it seems.
- We want to study PDEs on space which are:
  (1) Noncompact, complete manifolds with tame geometry at infinity like $\mathbb{R}^n$ and $\mathbb{H}^n$.
  (2) Singular spaces like cones, edges, stratified spaces, etc.
  (3) Spaces or operators which are degenerating; adiabatic limits, neck stretching, geometric gluing, etc.
- Examples of singular spaces: level sets of Morse functions; algebraic varieties; compactification of moduli spaces; compactification of locally symmetric spaces.
- Focus on elliptic operators: $L = \sum_{|\alpha| \leq m} a_\alpha(z) D_z^\alpha$. The principal symbol equals $\sigma_m(L)(z,\xi) = \sum_{|\alpha|=m} a_\alpha(z)\xi^\alpha$, and ellipticity means that this is invertible when $\xi \neq 0$.
- A priori estimates vs. parametrices:
  - The classical Sobolev estimates for an elliptic operator $\|u\|_{H^{s+m}} \leq (\|Lu\|_{H^s} + \|u\|_0)$ (strictly speaking, the norm on the left should be over a domain which is smaller than the one used on the right;
  - A parametrix for L is an approximate inverse, namely a pseudodifferential operator $G \in \Psi^{-m}$ such that $LG = I - R_1$, $GL = I - R_2$ where the two operators $R_1, R_2 \in \Psi^{-\infty}$ are 'residual'.

    Existence of a pseudodifferential parametrix and knowledge of its mapping properties imply the Sobolev estimates:

    - * $Lu = f$ implies $u - R_1 u = Gf$, which gives $\|u\|_{H^{s+m}} \leq C(\|f\|_{H^s} + \|u\|_0)$ since $G : H^s \to H^{s+m}$ is bounded and $R_1 : H^t \to H^{t'}$ is bounded for all t, t'

- One can use the a priori estimates plus a bit of functional analysis to deduce the existence of a (generalized) inverse for L, i.e., an inverse up to finite rank errors (the projects onto the kernel and cokernel). However, this does not tell you the structure of this generalized inverse.
- Our goal: a global theory of parametrices:
  - L a differential operator on manifold M, G a parametrix for L with $G(z, z') \in \mathcal{D}'(M \times M)$ its Schwartz kernel.
    - * We care about the geometric structure of G
  - After compactifying, things become interesting at the boundary
- Singular integral operators $\dfrac{F(\frac{z}{|z|})}{|z|^n}$, $\int_{S^{n-1}} F = 0$.
  - Oscillatory integral representation of ΨDOs
  - Melrose and collaborators led to Schwartz kernels
- Hadamard Parametrix Construction: (due to Friedlander)
  - Given L, find the parametrix G, $G \sim \sum_{j=0}^{\infty} G_j$ with $LG_0 = I - R_0$
  - If $L = \sum_{|\alpha|=m} a_\alpha(z_0) D_z^\alpha$; $G_0(z, \tilde{z}) = \mathcal{F}^{-1}\left(\dfrac{1}{\sum_{|\alpha|=m} a_\alpha(z_0)\xi^\alpha}\right)$
  - For χ a suitable cutoff function, this is

$$\int e^{i(z-\tilde{z})\cdot\xi} \frac{\chi(\xi)}{\sum_{|\alpha|=m} a_\alpha(z_0)\xi^\alpha} d\xi$$

  - For instance, if $m = 2$ and $L = \Delta$, then $\mathcal{F}^{-1}\left(\dfrac{1}{\xi^2}\right) = \dfrac{1}{|z - \tilde{z}|^{n-2}}$
  - $LG_0 = I +$ error and $G_0(z, \tilde{z}) \sim a_0(z) d(z, \tilde{z})^{m-n}$
  - $LG_0 = I - R_0(z, \tilde{z})$, $R_0(z, \tilde{z}) \sim d(z, \tilde{z})^{+n}$
  - $L(G_0 + ... + G_N) = I - R_N$, $R_N \sim d(z, \tilde{z})^{-n+N+1} + C^\infty(M \times M)$.
  - Want an asymptotic sum: $\tilde{G} \sim \sum G_j$, $L\tilde{G} = I - \tilde{R}$; $R \in C^\infty(M \times M)$.
  - In this construction, any M works but if M is open or singular, then $\tilde{R} \in C^\infty(M \times M)$ is not necessarily a compact operator.
  - $\tilde{G}^t L = I - \tilde{R}^t$, $\tilde{R}^t : \mathcal{E}'(M) \to C^\infty(M)$ does not improve growth.
- Turn our attention to $\mathbb{H}^n$:
  - Metric in half-space model:

$$\frac{dx^2 + dy^2}{x^2} \tag{1}$$

  - Metric in Poincaré disk:

$$\frac{4|dz|^2}{(1 - |z|^2)^2} \tag{2}$$

- Metric in Klein model:

$$\frac{4(\sum z_j dz_j^2)}{(1-|z|^2)^2} + \frac{4|dz|^2}{1-|z|^2} \tag{3}$$

- Laplacian in half-space:

$$\Delta_g = x^2\partial_x^2 + (2-n)x\partial_x + x^2\Delta_y \tag{4}$$

which degenerates at $x = 0$.
- Laplacian in Poincaré:

$$\Delta_g = (1-|z|^2)^2\Delta_z - 2(2-n)\sum z_i\partial_{z_i} \tag{5}$$

- (M,g) conformally compact with $g = \rho^{-2}\bar{g}$, $\rho = 0$ on ∂M, $d\rho \neq 0$.
- (r,θ) on $\mathbb{H}^n$ with r distance from origin and $\theta \in S^{n-1}$,

$$g = dr^2 + \sinh^2 r d\theta^2 \tag{6}$$

and

$$\Delta = \sinh^{1-n} r\partial_r(\sinh^{n-1} r\partial_r) + \frac{1}{\sinh^2 r}\Delta_\theta \tag{7}$$

- Want to solve

$$G'' + (n-1)\frac{\cosh r}{\sinh r}G' = 0,\ r > 0 \tag{8}$$

- Thus, either $G \sim r^0, r^{2-n}$
- With $\rho = e^{-r}$, we have $G \sim \rho^{n-1}$ as $\rho \to 0$, i.e., $r \to \infty$.
- At the boundary of $M \times M$ compactified we have nice expansions with $G \sim r^{2-n}$ toward the diagonal.
- What happens at the corners of the diagonal?
- Blow up the corner along the diagonal
- Dilation invariant: $G(x,y,\tilde{x},\tilde{y}) = G(\lambda x, \lambda y, \lambda\tilde{x}, \lambda\tilde{y})$
- How to localize?
- $\mathbb{H}^n \setminus \Omega$, $\Delta + V$, $V \in C_0^\infty$, define $\tilde{G} = \tilde{\chi}_1 G_{in}\chi_1 + \tilde{\chi}_2 G_{out}\chi_2$ where $\chi_1, \tilde{\chi}_1$ compactly supported in U_1 and $\chi_2, \tilde{\chi}_2$ compactly supported in U_2 where $\Omega \subset U_1 \subset U_2$.
- $G_{out} = G_{\mathbb{H}^n}$, G_{in} =standard local ΨDO.
- Want to obtain local parametrices and glue them together.