MSRI LECTURES ON NONLINEAR WAVES

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ABSTRACT. Rough notes for lectures on nonlinear waves at the MSRI introductory workshop in Fall 2019.

- Local Theory for Linear Waves:
  - Lorentzian metric $g = -dt^2 + dx^2$ on $\mathbb{R}^t \times \mathbb{R}_x^{n-1}$
  - Given $v \in T_p(\mathbb{R}^t \times \mathbb{R}_x^{n-1})$
    - We say v is timelike if $g(v, v) < 0$.
    - We say v is null/lightlike if $g(v, v) = 0$.
    - We say v is spacelike if $g(v, v) > 0$.
  - $\frac{\partial}{\partial t}$ is timelike, $\frac{\partial}{\partial t} + \frac{\partial}{\partial x_1}$ is null.
  - Naturally associated to g is the wave operator (Laplace-Beltrami operator)

$$\Box_g = -\partial_t^2 + \Delta_x.$$

  - Consider the forcing problem

$$\begin{cases} \Box_g u = f \\ u|_{t<0} = 0. \end{cases} \tag{1}$$

    where $\text{supp}(f)$ is bounded in $\mathbb{R}_x^{n-1}$.
    - There exists a unique solution $u(t, x)$
  - Properties of solution:
    - Finite speed of propagation: a bound on the support of u
      - $\text{supp} u_t$ contained in the causal future of $\text{supp}(f)$
      - This allows us to localize
    - Regularity: obtain a basic estimate
      - $\|\chi u\|_{H^s} \lesssim \|f\|_{H^{s-1}}$ where χ is smooth with compact support in $\mathbb{R}^t \times \mathbb{R}_x^{n-1}$.
      - The solution gains one derivative
    - Sketch of regularity proof:
      - Suppsoe $f \in C^\infty$.
      - We already know $u \in C^\infty$ for $t < 0$.
      - Idea: propagate the regularity.
      - $G(t, x, \sigma, \xi) = \sigma_p(\Box_g) = -\sigma^2 + |\xi|^2$.

- · Outside of $\Sigma_\Box = \{(t,x,\sigma,\xi) : |\sigma| = |\xi|\}$ we have elliptic estimates.
- · Outside of $\Sigma_\Box$, we need to propagate estimates.
- · The Hamiltonian vector field: $H_G = -2\sigma\partial_t + 2\xi\partial_x$
- · The integral curves of H_G in $\Sigma_\Box$ are null geodesics for g
- · So, by propagation of singularities we have

$$\|\chi u\|_{H^s} \lesssim \|B_1 u\|_{H^s} + \|B_2 \Box_g u\|_{H^{s-1}} + \text{error}$$

  where χ is supported in $t > 0$ and B_1 supported in $t < 0$ and intersects the backwards light cone of the support of χ
- · Thus we have the estimate we desired

- Global Theory for Linear Waves:
  - Consider De Sitter space: $g_0 = \dfrac{-d\tau^2 + dy^2}{\tau^2}$ on $M = [0,\infty)_\tau \times \mathbb{R}^{n-1}_y$
    - * Think of $\tau = e^{-t_*}$ where t_* is the usual time function.
    - * Look at light cone of $(0,0)$, $\tau = |y|$, and consider the forcing problem again with $\text{supp}(f)$ contained in that light cone
    - * The metric is singular at $(0,0)$, so blow up the manifold by introducing the coordinate $x = \dfrac{y}{\tau}$.
    - * Let Ω be a domain containing a portion of the light cone in the blown up manifold and consider the forcing problem in Ω.
  - After blowup, the metric becomes

$$g_0 = -(1-x^2)\frac{d\tau^2}{\tau^2} + 2x\,dx\frac{d\tau}{\tau} + dx^2$$

    which is a b-metric
  - $\Box_{g_0} \in \text{Diff}^2_b(\Omega)$, $\Omega = [0,1)_\tau \times X$, $X = \{|x| < 2\}$
  - $\Box_{g_0} = -(\tau\partial_\tau)^2 + 2\tau\partial_\tau x\partial_x + (1-x^2)\partial_x^2 + \text{l.o.t.}$
  - Observe that $\Box_{g_0}$ is dilation invariant in τ (corresponds to translation invariance in t_*
  - Led to taking Fourier transform in τ, i.e., $\tau\partial_\tau$ becomes σ
  - $\Box_{g_0} \to \widehat{\Box_{g_0}}(\sigma) \in \text{Diff}^2(X)$
  -

$$u(\tau,x) = \frac{1}{2\pi}\int \tau^{i\sigma}\widehat{\Box_{g_0}}(\sigma)^{-1}\hat{f}(\sigma)d\sigma$$

  - Need to understand if $\widehat{\Box_{g_0}}(\sigma)^{-1}$ is analytic/meromorphic and where poles are if they exist
- Poles: resonances of $\widehat{\Box_{g_0}}(\sigma)^{-1}$
  - Poles along the imaginary axis with imaginary part less than or equal to zero

  - $u(\tau, x) = u_0 + \tilde{u}(\tau, x)$ where $u_0 \in \mathbb{R}$ and $|\tilde{u}| \leq C\tau$ which means you can differentiate with respect to $\tau\partial_\tau$ as much as we want
  - Really, $u \in \mathscr{A}^{\mathcal{E}}_{phg}(\Omega)$ where $\mathcal{E} = \{\text{imaginary resonances}\}$
- Qualitatively: $|u_0| + \|\tilde{u}\|_{\tau H^s_b} \lesssim \|f\|_{\tau H^{s-1}_b}$
- Nonlinear Theorem: (H-Vasy) Consider

$$\begin{cases} \Box_{g(x,u)} u = f \\ u|_{\tau>1} = 0 \end{cases} \tag{2}$$

where $g^{ij}(x,u) = g^{ij}_0(x) + c^{ij}|\nabla u|^2_{g_0}$. For small f in τH^{s-1}_b, the forcing problem has a global solution in Ω and $u \in \mathbb{R} \oplus \tau H^s_b$
  - Idea of proof:
    * Iteration scheme
    * $\Box_{g_0} u^{(0)} = f$ can be solved as before
    * Iterate: $\Box_{g(x,u^{(k)})} u^{(k+1)} = f$
    * Solution: $u = \lim_{k\to\infty} u^{(k)}$
    * How does this work?
    * If $u^{(k)} \in \mathbb{R} \oplus \tau H^s_b$, then $\Box_{g(x,u^{(k)})} = \Box_{g_0} + O(\tau)$
    * Need to show that for such $u^{(k)}$, $(\Box_{g(x,u^{(k)})})^{-1} : \tau H^{s-1}_b \to \mathbb{R} \oplus \tau H^s_b$
    * To do this, we need
      (1) Regularity and asymptotics for $\Box_g^{-1}$, $g = g_0 + O(\tau)$ smooth
      (2) Do the previous for finite regularity metrics
    * For (1), do the following:
      · Analyze regularity, not decay, of $\Box_g u = f$ microlocally
      · Use $\Box_{g_0}$ to get precise asymptotics
    * For the first part, $\Box_g = G(\tau, x, \tau\partial_\tau, \partial_x) = \mathrm{Op}_b(G(\tau, x, \sigma, \xi))$ where G is in $S^2({}^bT^*\Omega)$
    * The Hamiltonian vector field is a vector field on ${}^bT^*\Omega$, $\partial_\sigma, \partial_\xi, \tau\partial_\tau, \partial_x$, and is C^∞ up to $\tau = 0$
    * $H_G = H_{G_0} + O(\tau)$
    * The blowup procedure spreads high frequency waves into lower frequency waves
    * Upshot: radial point estimates; $\|u\|_{\tau^\alpha H^s_b} \lesssim \|\chi u\|_{H^s} + \|f\|_{\tau^\alpha H^{s-1}_b}$
    * If we slide the support of χ to $\tau = 10$ for instance, then we obtain $\|u\|_{\tau^\alpha H^s_b} \lesssim \|f\|_{\tau^\alpha H^{s-1}_b} + \text{error}$ for $s > \frac{1}{2} + \alpha$
    * Energy estimates yield $\|u\|_{\tau^\alpha L^2} \lesssim \|f\|_{\tau^\alpha L^2}$
    * For the second part of (1), we want to understand $\Box_g u = f$ which is equivalent to $\Box_{g_0} u = f - (\Box_g - \Box_{g_0})u$ where the second term decays like $\tau^{\alpha+1}$
    * We use $\Box_{g_0}^{-1}$ to get $u = O(\tau^{\alpha+1})$ if $f \in C^\infty_c$

* Iterate, get $u = u_0 + \tilde{u}$, $\tilde{u} = O(\tau)$ which ultimately yields $(\Box_{g(x,u^{(k)})})^{-1} : \tau^\alpha H_b^{s-1} \to \mathbb{R} \oplus \tau H_b^s$