

NOTETAKER CHECKLIST FORM

(Complete one for each talk.)

Name: Arun Debray Email/Phone: a.debray@math.utexas.edu
Speaker's Name: Sarah Witherspoon
Talk Title: Hopf Algebras I
Date: 01/27/20 Time: 09:30 am / pm (circle one)

Please summarize the lecture in 5 or fewer sentences: She provided most of the definition of Hopf algebras, and discussed several examples: the "classical" examples of $k[G]$ and $U(\mathfrak{g})$, as well as quantum S_2 and small quantum S_2 . Then discussed the question: what structures can we recover on the category of modules over a Hopf algebra?

CHECK LIST

(This is **NOT** optional, we will **not** pay for incomplete forms)

- Introduce yourself to the speaker prior to the talk. Tell them that you will be the note taker, and that you will need to make copies of their notes and materials, if any.
- Obtain ALL presentation materials from speaker. This can be done before the talk is to begin or after the talk; please make arrangements with the speaker as to when you can do this. You may scan and send materials as a .pdf to yourself using the scanner on the 3rd floor.
 - **Computer Presentations:** Obtain a copy of their presentation
 - **Overhead:** Obtain a copy or use the originals and scan them
 - **Blackboard:** Take blackboard notes in black or blue **PEN**. We will **NOT** accept notes in pencil or in colored ink other than black or blue.
 - **Handouts:** Obtain copies of and scan all handouts
- For each talk, all materials must be saved in a single .pdf and named according to the naming convention on the "Materials Received" check list. To do this, compile all materials for a specific talk into one stack with this completed sheet on top and insert face up into the tray on the top of the scanner. Proceed to scan and email the file to yourself. Do this for the materials from each talk.
- When you have emailed all files to yourself, please save and re-name each file according to the naming convention listed below the talk title on the "Materials Received" check list.
(YYYY.MM.DD.TIME.SpeakerLastName)
- Email the re-named files to notes@msri.org with the workshop name and your name in the subject line.

SARAH WITHERSPOON: HOPF ALGEBRAS, I

Our perspective on Hopf algebras, their actions on rings and modules, and the structures on their categories of rings and modules, will be to think of them as generalizations of group actions and representations; groups actions are symmetries in the usual sense, and Hopf algebra actions are often related to “quantum symmetries.”

We’re not going to give the full definition of a Hopf algebra, because it would require drawing a lot of commutative diagrams, but we’ll say enough to give the picture.

Throughout this talk we work over a field k ; all tensor products are of k -vector spaces.

Definition 0.1. A *Hopf algebra* is an algebra A together with k -linear maps $\Delta: A \rightarrow A \otimes A$, called *comultiplication*; $\varepsilon: A \rightarrow k$, called the *counit*; and $S: A \rightarrow A$, called the *coinverse*. These maps must satisfy some properties, including that ε is an algebra homomorphism and that S is an *anti-automorphism*, i.e. that $S(xy) = S(y)S(x)$.

The definition is best understood through examples.

Example 0.2.

- (1) Let G be a group. Then the group algebra $k[G]$ is a Hopf algebra, where for all $g \in G$, $\Delta(g) := g \otimes g$, $\varepsilon(g) := 1$, and $S(g) := g^{-1}$. This is a key example that allows us to generalize ideas from group actions to Hopf algebra actions: whenever we define a notion for Hopf algebras, when we implement it for $k[G]$ it should recover that notion for groups.
- (2) Let \mathfrak{g} be a Lie algebra over k . Then its universal enveloping algebra $\mathcal{U}(\mathfrak{g})$ is a Hopf algebra, where for all $x \in \mathfrak{g}$, $\Delta(x) := x \otimes 1 + 1 \otimes x$, $\varepsilon(x) := 0$, and $S(x) := -x$. Since ε is an algebra homomorphism, $\varepsilon(1_{\mathcal{U}(\mathfrak{g})}) = 1$.

For example,

$$(0.3) \quad \mathcal{U}(\mathfrak{sl}_2) = k\langle e, f, h \mid ef - fe = h, he - eh = 2e, hf - fh = -2f \rangle,$$

given explicitly by the basis of \mathfrak{sl}_2

$$(0.4) \quad e := \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix}, \quad f := \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix}, \quad h := \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}. \quad \blacktriangleleft$$

Both of these examples are classical, in that they’ve been known for a long time. But more recently, in the 1980s, people discovered new examples, coming from quantum groups.

Example 0.5 (Quantum \mathfrak{sl}_2). Let $q \in k^\times \setminus \{\pm 1\}$. Then, given a simple Lie algebra \mathfrak{g} , we can define a “quantum group,” $\mathcal{U}_q(\mathfrak{g})$, which is a Hopf algebra. For example, for \mathfrak{sl}_2 ,

$$(0.6) \quad \mathcal{U}_q(\mathfrak{sl}_2) = k\left\langle E, F, K^{\pm 1} \mid EF - FE = \frac{K - K^{-1}}{q - q^{-1}}, KE = q^2EK, KF = q^{-2}EK \right\rangle,$$

with comultiplication

$$(0.7a) \quad \Delta(E) := E \otimes 1 + K \otimes E$$

$$(0.7b) \quad \Delta(F) := F \otimes K^{-1} + 1 \otimes F$$

$$(0.7c) \quad \Delta(K^{\pm 1}) := K^{\pm 1} \otimes K^{\pm 1}$$

and counit $\varepsilon(E) = \varepsilon(F) = 0$ and $\varepsilon(K) = 1$. This generalizes to other simple \mathfrak{g} , albeit with more elaborate data. \blacktriangleleft

Example 0.8 (Small quantum \mathfrak{sl}_2). Let q be an n^{th} root of unity. Then, as before, given a simple Lie algebra \mathfrak{g} , we can define a Hopf algebra $u_q(\mathfrak{g})$, called the *small quantum group* for \mathfrak{g} and q , which is a finite-dimensional vector space over k ; for \mathfrak{sl}_2 , this is

$$(0.9) \quad u_q(\mathfrak{sl}_2) = \mathcal{U}_q(\mathfrak{sl}_2) / (E^n, F^n, K^n - 1). \quad \blacktriangleleft$$

Before we continue, we need some useful notation for comultiplication, called *Sweedler notation*. Let A be a Hopf algebra and $a \in A$; then we can symbolically write

$$(0.10) \quad \Delta(a) = \sum_{(a)} a_1 \otimes a_2.$$

Comultiplication in a Hopf algebra is *coassociative*, in that as maps $A \rightarrow A \otimes A \otimes A$,

$$(0.11) \quad (\Delta \otimes \text{id}) \circ \Delta = (\text{id} \otimes \Delta) \circ \Delta.$$

Therefore when we iterate comultiplication, we can symbolically write

$$(0.12) \quad (\text{id} \otimes \Delta) \circ \Delta(a) = \sum_{(a)} a_1 \otimes a_2 \otimes a_3$$

without worrying about parentheses.

Actions on rings. Hopf algebra actions on rings generalize group actions on rings by automorphisms and actions of Lie algebras on rings by derivations. If a group G acts on a ring R , then for all $g \in G$ and $r, r' \in R$,

$$(0.13a) \quad g(rr') = (gr)(gr')$$

$$(0.13b) \quad g(1_R) = 1_R.$$

In $k[G]$, our Hopf algebra avatar of G , $\Delta(g) = g \otimes g$, and $\varepsilon(g) = 1$.

If a Lie algebra \mathfrak{g} acts on a ring R by derivations, then for all $x \in \mathfrak{g}$ and $r, r' \in R$,

$$(0.14a) \quad x \cdot (rr') = (x \cdot r)r' + r(x \cdot r')$$

$$(0.14b) \quad x \cdot (1_R) = 0.$$

In $\mathcal{U}(\mathfrak{g})$, our Hopf algebra avatar of \mathfrak{g} , $\Delta(x) = x \otimes 1 + 1 \otimes x$, and $\varepsilon(x) = 0$. These two examples suggest how we should implement a general Hopf algebra action on a ring: comultiplication tells us how to act on the product of two elements, and the counit tells us how to act on 1.

Definition 0.15. Let A be a Hopf algebra and R be a k -algebra. An A -module algebra structure on R is data of an A -module structure on R such that for all $a \in A$ and $r, r' \in R$,

$$(0.16a) \quad a \cdot (rr') = \sum_{(a)} (a_1 \cdot r)(a_2 \cdot r')$$

$$(0.16b) \quad a \cdot (1_R) = \varepsilon(a)1_R.$$

Thus a group action as in (0.13) defines an action of the Hopf algebra $k[G]$, and a Lie algebra action as in (0.14) defines an action of the Hopf algebra $\mathcal{U}(\mathfrak{g})$.

Example 0.17. The quantum analogue of the \mathfrak{sl}_2 -action on $k[x, y]$, thought of as (functions on the) plane, there is an action of $\mathcal{U}_q(\mathfrak{sl}_2)$ on the *quantum plane*

$$(0.18) \quad R := k\langle x, y \mid xy = qyx \rangle.$$

This is a deformation of $k[x, y]$, which is the case $q = 1$. The explicit data of the action is

$$(0.19) \quad E \cdot x = 0 \quad F \cdot x = y \quad K^{\pm 1} \cdot x = q^{\pm 1}x$$

$$(0.20) \quad E \cdot y = x \quad F \cdot y = 0 \quad K^{\pm 1} \cdot y = q^{\mp 1}y.$$

One has to check that this extends to an action satisfying Definition 0.15, but it does, and R is an A -module algebra. Here E and F act as *skew-derivations*, e.g.

$$(0.21) \quad E \cdot (rr') = (E \cdot r)r' + (K \cdot r)(E \cdot r')$$

for all $r, r' \in R$. ◀

Given a Hopf algebra action of A on R in this sense, we can construct two useful rings: the *invariant subring*

$$(0.22) \quad R^A := \{r \in R \mid a \cdot r = \varepsilon(a) \cdot r \text{ for all } a \in A\},$$

and the *smash product ring* $R \# A$, which as a vector space is $R \otimes A$, with multiplication given by

$$(0.23) \quad (r \otimes a)(r' \otimes a') := \sum_{(a)} r(a_1 \cdot r') \otimes a_2 a'.$$

The smash product ring knows the A -module algebra structure on R . Often, rings we're interested in for other reasons are smash product rings of interesting Hopf algebra actions, and identifying this structure is useful.

Example 0.24. The *Borel subalgebra* of $\mathcal{U}_q(\mathfrak{sl}_2)$ is $k\langle E, K^{\pm 1} \mid KE = q^{-2}K \rangle$. This is isomorphic to the smash product $k[E] \# k\langle K \rangle$, where $k\langle K \rangle$ is the group algebra of the free group on the single generator K .

In fact, there's a sense in which $\mathcal{U}_q(\mathfrak{sl}_2)$ is a deformation of $k[E, F] \# k\langle K \rangle$: in this smash product ring, E and F commute, and we deform this to $\mathcal{U}_q(\mathfrak{sl}_2)$, in which they don't commute. \blacktriangleleft

Modules. Given a Hopf algebra A , what is the structure of its category of modules? The first thing we can do is take the tensor product of A -modules U and V using comultiplication: for $a \in A$, $u \in U$, and $v \in V$,

$$(0.25) \quad a \cdot (u \otimes v) = \sum_{(a)} a_1 \cdot u \otimes a_2 \cdot v.$$

Moreover, k has a canonical A -module structure via the counit: $a \cdot x := \varepsilon(a)x$ for $a \in A$ and $x \in k$. Finally, if U is an A -module, its vector space dual $U^* := \text{Hom}_k(U, k)$ has an A -module structure via S : for all $a \in A$, $u \in U$, and $f \in U^*$, $(a \cdot f)(u) := f(S(a)u)$.

The existence of tensor products, duals, and the ground field in the world of Hopf algebra modules is a nice feature: these aren't always present for a general associative algebra. Moreover, these constructions interact well with each other.

- (1) Coassociativity of Δ implies the tensor product is associative: for A -modules U , V , and W , we have a natural isomorphism $U \otimes (V \otimes W) \xrightarrow{\cong} (U \otimes V) \otimes W$.
- (2) In any Hopf algebra A , we have the condition

$$(0.26) \quad \sum_{(a)} \varepsilon(a_1)a_2 = \sum_{(a)} a_1\varepsilon(a_2)$$

for any $a_1, a_2 \in A$. This implies k , as an A -module, is the unit for the tensor product: we have natural isomorphisms $k \otimes U \cong U \cong U \otimes k$ for an A -module U .

- (3) Suppose U is an A -module which is a finite-dimensional k -vector space. Then it comes with data of a *coevaluation map* $c: k \rightarrow U \otimes U^*$ sending

$$(0.27) \quad 1 \mapsto \sum_i u_i \otimes u_i^*,$$

where $\{u_i\}$ is a basis for U over k and $\{u_i^*\}$ is its dual basis; this map turns out to be independent of basis. We also have an *evaluation map* $e: U^* \otimes U \rightarrow k$ sending $f \otimes u \mapsto f(u)$. Now, not only are these A -module homomorphisms, but the composition

$$(0.28) \quad U \xrightarrow{c \otimes \text{id}_U} U \otimes U^* \otimes U \xrightarrow{\text{id}_U \otimes e} U$$

is the identity map.

Definition 0.29. A *tensor category*, or *monoidal category* is a category \mathcal{C} together with a functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, an object $\mathbf{1} \in \mathcal{C}$ called the *unit*, and natural isomorphisms $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$ and $\mathbf{1} \otimes U \cong U \cong U \otimes \mathbf{1}$ for all objects U , V , and W in \mathcal{C} , subject to some coherence conditions.

Our key examples of tensor categories are the category of modules over a Hopf algebra A , as well as the subcategory of finite-dimensional modules.

If the coinverse of A is invertible, which is always the case when A is finite-dimensional over k , then $\mathcal{C} = \text{Mod}_A$ is a *rigid* tensor category, meaning that every object U has a *right dual* ${}^*U := \text{Hom}_k(U, k)$, which means the composition (0.28) is the identity.

Remark 0.30. Notations for left and right duals differ. We're following [EGNO15], but Bakalov-Kirillov [BK01] use a different convention; be careful! ◀

Some Hopf algebras' categories of modules have additional structure or properties: they might be semisimple, or braided, or even symmetric. This amounts to additional information on the Hopf algebra itself.

REFERENCES

- [BK01] Bojko Bakalov and Alexander Kirillov, Jr. *Lectures on tensor categories and modular functors*, volume 21 of *University Lecture Series*. American Mathematical Society, Providence, RI, 2001. 4
- [EGNO15] Pavel Etingof, Shlomo Gelaki, Dmitri Nikshych, and Victor Ostrik. *Tensor categories*, volume 205 of *Mathematical Surveys and Monographs*. American Mathematical Society, Providence, RI, 2015. 4

Hopf Algebras I

Thank organizers, MRI - just to see so many people here at the opening of the spring program

Some motivating remarks about symmetry groups & quantum groups. Hopf algs (part of quantum groups)

Defn A Hopf algebra is an algebra A (over a field k) together with k -linear maps

$$\otimes = \otimes_k$$

$$\Delta: A \rightarrow A \otimes A, \quad \varepsilon: A \rightarrow k, \quad S: A \rightarrow A$$

satisfying some properties (that define properties of groups - can look them up)

or other convenient ring

Examples (1) $A = kG$, group algebra, $\forall g \in G$:

$$\Delta(g) = g \otimes g, \quad \varepsilon(g) = 1, \quad S(g) = g^{-1}$$

(2) $A = U(\mathfrak{g})$, univ. env. alg. of Lie algebra \mathfrak{g} , $\forall x \in \mathfrak{g}$:

$$\Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0, \quad S(x) = -x$$

e.g. $U(\mathfrak{sl}_2) \cong k \langle e, f, h \mid ef - fe = h, he - eh = 2e, hf - fh = -2f \rangle$

$$\mathfrak{sl}_2 = 2 \times 2 \text{ matrices of det } 1 \quad e \leftrightarrow \begin{pmatrix} 0 & 1 \\ 0 & 0 \end{pmatrix} \quad f \leftrightarrow \begin{pmatrix} 0 & 0 \\ 1 & 0 \end{pmatrix} \quad h \leftrightarrow \begin{pmatrix} 1 & 0 \\ 0 & -1 \end{pmatrix}$$

(3) $A = U_\hbar(\mathfrak{g})$, quantum group $\mathfrak{g} \in k^\times, \hbar \neq \pm 1$

$$\text{e.g. } U_\hbar(\mathfrak{sl}_2) = k \langle E, F, K^{\pm 1} \mid EF - FE = \frac{K - K^{-1}}{\hbar}, KE = \hbar^2 EK, KF = \hbar^{-2} FK \rangle$$

$$\Delta(E) = E \otimes 1 + K \otimes E, \quad \Delta(F) = F \otimes K^{-1} + 1 \otimes F, \quad \Delta(K^{\pm 1}) = K^{\pm 1} \otimes K^{\pm 1}$$

(4) $A = u_\hbar(\mathfrak{g})$, small quantum group, $\hbar^n = 1$

$$U_\hbar(\mathfrak{g}) / (E^n, F^n, K^n - 1)$$

(finite dim as v.s., generating fn's)

Sweedler notation: $a \in A$

(one of properties of Δ is coassociativity)

$$\Delta(a) = \sum_{(a)} a_1 \otimes a_2 \quad \text{or simply } a_1 \otimes a_2 \text{ (symbolically)}$$

$$\sum_{(a)} \sum_{(a_1)} \sum_{(a_2)} \sum_{(a_3)} a_1 \otimes a_2 \otimes a_3 = \sum_{(a)} a_1 \otimes a_2 \otimes a_3$$

quick example

Actions on rings (see Susan Montgomery's book)

Groups act by automorphisms: $\text{group } G \text{ acts on ring } R$

$$g \cdot (r r') = (g \cdot r)(g \cdot r'), \quad g \cdot 1_R = 1_R \quad \forall g \in G, r, r' \in R$$

$$\text{cf } \Delta g = g \otimes g \quad \varepsilon(g) = 1$$

(symmetry groups)

group gradings correspond to action of dual of group algebra

Lie algebras act by derivations: $\text{Lie alg } \mathfrak{g} \text{ acts on ring } R$

$$x \cdot (r r') = (x \cdot r) r' + r (x \cdot r'), \quad x(1_R) = 0 \quad \forall x \in \mathfrak{g}, r, r' \in R$$

$$\text{cf. } \Delta(x) = x \otimes 1 + 1 \otimes x, \quad \varepsilon(x) = 0$$

More generally:

Defn let A be a Hopf algebra and let R be a k -algebra.

R is an A -module algebra if

- (i) R is an A -module
- (ii) $a \cdot (r r') = \sum (a_{(1)} \cdot r) (a_{(2)} \cdot r')$ $\forall a \in A, r, r' \in R$
- (iii) $a \cdot (1_R) = \varepsilon(a) \cdot 1_R$ $\forall a \in A$

Example $A = U_q(\mathfrak{sl}_2)$ $R = k\langle x, y \mid xy - qyx \rangle$ (quantum plane)

$E \cdot x = 0$	$F \cdot x = y$	$K \cdot x = qx$	
$E \cdot y = x$	$F \cdot y = 0$	$K \cdot y = q^{-1}y$	(ideal gen by $xy - qyx$ preserved)

K acts as an automorphism and (E, F) act as skew derivations

$$\text{cf. } E(r r') = (E \cdot r) r' + (K \cdot r) (E \cdot r') \quad \forall r, r' \in R$$

Discuss analogy between symmetry groups and actions of Hopf algebras \leadsto quantum symmetry

alg geom and NC geom, actions on function algebras on (NC) spaces

Modules

Let A be a Hopf algebra, let U, V be A -modules. (left)

$U \otimes V$ is an A -module via Δ :

$$a \cdot (u \otimes v) := \sum_{(a)} (a_1 \cdot u) \otimes (a_2 \cdot v) \quad \forall a \in A, u \in U, v \in V$$

k is an A -module via ε :

$$a \cdot c = \varepsilon(a)c \quad \forall a \in A, c \in k$$

left dual $U^* := \text{Hom}_k(U, k)$ is an A -module via S :
(EGNO) (k-lin. fns from $A \times k$)

$$(a \cdot f)(u) = f(S(a) \cdot u) \quad \forall a \in A, u \in U, f \in \text{Hom}_k(U, k)$$

right dual $*U = \text{Hom}_k(U, k)$ is an A -module via S^{-1} when S is invertible: (which it always is for f.d. Hopf algs)
(EGNO) $(a \cdot f)(u) = f(S^{-1}(a) \cdot u)$ $\forall A\text{-mods } U, V, W$

Properties (i) $U \otimes (V \otimes W) \cong (U \otimes V) \otimes W$ (by associativity)

(ii) $k \otimes U \cong U \cong U \otimes k$ $\forall A\text{-mods } U$ (by another property: $\sum \varepsilon(a_1)a_2 = a = \sum a_1 \varepsilon(a_2) \forall a \in A$)

[5 of [1] p. 100/101]

(iii) and a property for U^* (rigidity) for U f.d.,

$$\text{Hom}_k(U, U) \xrightarrow{\text{coev}_U \otimes \text{id}_U} U \otimes U^* \otimes U \xrightarrow{\text{id}_U \otimes \text{ev}_U} U$$

Composition is id. map on U

$$\text{coev}_U(1) = \sum_i u_i \otimes u_i^*$$

\leftarrow basis, dual basis

$$\text{ev}_U(f \otimes u) = f(u)$$

Defn A tensor category (or monoidal category) is a category \mathcal{C} together with a functor $\otimes: \mathcal{C} \times \mathcal{C} \rightarrow \mathcal{C}$, an object $\mathbb{1}$, and natural isomorphisms

$$U \otimes (V \otimes W) \cong (U \otimes V) \otimes W, \quad \mathbb{1} \otimes U \cong U \cong U \otimes \mathbb{1} \text{ for all objects } U, V, W$$

satisfying some properties ((triangle and pentagon axioms)).

(rigid if left and right duals exist)

Example The category of modules of a Hopf algebra A is a tensor category. (rigid when S invertible)
(f.d.)

(sometimes w/ additional properties such as a braiding)

MSRI Intro. Workshop
 Wed, Jan 29, 2020
 9:30 - 10:30 am

Hopf Algebras II

- We've been hearing a lot about (co)tensor algs of various types in other talks, such as Victor Ostrik & Eric Powell
- I'm most interested in non-s.s. ones, particularly those that are mod of non-i.c. Hopf algebras such as small q 's (we saw in V.O.'s lecture that on contract a r.s. cat to, but -
- How to understand - may be wild type etc
- Connect to classification results of various types (cf other talk) - classify some types of tensor cats \leftrightarrow classify mod of Hopf algs \leftrightarrow classify Hopf algs

Motivating Problems

- Classify some types of tensor categories
- Classify some types of Hopf algebras
- Understand objects in tensor categories, modules of Hopf algebras

Hopf algebra cohomology

Defn. An n -extension of A -modules U, V is a short exact sequence of A -module homs:

$$0 \rightarrow V \rightarrow M_n \rightarrow \dots \rightarrow M_2 \rightarrow M_1 \rightarrow U \rightarrow 0$$

(i.e. A mod homs, image of each map = kernel of next)

(non-s.s. case: an exact seq does not always split so you want to understand mod via how they're put together in n -ext)

Defn Hopf algebra cohomology: $H^n(A, k) := \text{Ext}_A^n(k, k)$ all n -extensions modulo an equivalence relation (describe) - OTHER WAYS TO DEFINE IT!

Yoneda splice m -ext, n -ext \rightsquigarrow $(m+n)$ -ext

$$0 \rightarrow k \rightarrow M_m \rightarrow \dots \rightarrow M_1 \rightarrow k \rightarrow 0, \quad 0 \rightarrow k \rightarrow N_n \rightarrow \dots \rightarrow N_1 \rightarrow k \rightarrow 0$$

(can check exact)

Cohomology ring $H^*(A, k) := \bigoplus_{n \geq 0} H^n(A, k)$
 under Yoneda splice, is a graded ring
 (in fact, it is graded commutative by the Eckmann-Hilton argument)
 OTHER WAYS TO DEFINE THIS PRODUCT!

More generally let \mathcal{C} be a tensor category
 $H^*(\mathcal{C}, \mathbb{1})$ defined similarly, as a graded commutative ring.

Conjecture (Friedlander-Mislin '97, Etingof-ostrik '04, ...)
 Let A be a f.d. Hopf algebra. Then $H^*(A, k)$ is finitely generated.

(I am not defining finite tensor cats)

More generally let \mathcal{C} be a finite tensor category. Then $H^*(\mathcal{C}, \mathbb{1})$ is fin. gen.
 and $\text{Ext}_A^*(U, V) \cong \text{Ext}_{\mathcal{C}}^*(X, Y)$ is f.g. as mod over $H^*(A, k) \cong H^*(\mathcal{C}, \mathbb{1})$

Why is this important:

(e.g. Reineh-Plavnik-W on arXiv 19)

There is a theory of "varieties for modules" that is most useful when conjecture is true. (can say more about later - but for now just take as motivation)

e.g. recent work w/ Peter Reineh, Julia Plavnik on support vars for fdc's on arXiv and earlier work w/ Julia Plavnik

Warning: An analogous conjecture for Hoch. Coh. of fin. dim. alg. is false.

20th century results: Conjecture is true for

(1) $A = kG$, G a finite group, char $k = p > 0$ (o.w. trivial by Maschke's Thm) (Golod; Venkov; Evens ~1960) Evens proof outline - reduces to G a p -group, $|G| = p^n$ - induction on n : $2 < 2(G)$, $|Z| = p$ LHS S.S. for $Z \rightarrow G \rightarrow G/Z$

(p. 132 Benson II)

(2) A - restricted env. alg., char $k = p > 0$ (f.d. quotient of $U(\mathfrak{g})$) (Friedlander - Parshall 1983) (see Oberwolfach talk notes 2018 for summary of proof)

(3) $A = u_{\mathfrak{g}}(\mathfrak{g})$, small quantum group (char $k = 0$) (Ginzburg - Kawan 1993) (odd $n \geq 3$ or 4) (antipode parameter)

(i.e. comultiplication same if order swapped)

(4) A - f.d. co-commut. Hopf alg., char $k = p > 0$ (includes (1) + (2), not (3)) (Friedlander - Suslin 1997)

* "like all major historical events, I remember exactly where I was."

- Why can't we just prove it in general? - well, back to classification! We just don't yet know enough about Hopf algebras / tensor categories generally!



21st century results

(see p. 4 of Oberwolfach notes)

in many many more types

- Gordon '00 (fin. results of g-fen alg at mod of 4) not div by 2, 3, 5, 7
- Mestrele - Pert Svan - Schauenberg - W '10 (ptd. unlabel gfs, restriction on primes dividing order)
- Beutel - Natori - Parshel - Pillemer '14 (removed some restrictions on parameter in GK '93 (k=6))
- Nguyen - W '14 (some skew g-fen in per char)
- Drupieski '16 (fin. syngp schemes)
- Vay - Stefan '16 - Form in Euler alpha
- Friedberka - Negrin '18 - D.d.'s of commut Aylt alg
- Nguyen - Wang - W $\cdot \dim p^2$, more general results - 2 pgs APOLOGIES IF I HAVE INADVERTENTLY MISSED SOMEONE
- Erdmann - Solberg - Wang
- Negrin - Plamnik - some results for f.t. c.s. (quality and cent p) (perhaps the 1st paper w/ results in this generality)

(2 papers with Van Nguyen Xingting Wang)

(Also works of Benson, Etingof, Orlik constructing new unass. f.t.c.s in per char.)
not on chromlogy - but -

- Lots ongoing projects -

ONE IS MY WORK w/ Nicolas Androukiewitch,
Ivan Ariano, and Julia Petrova
(Nicolas & Julia will be here at MPI for a couple of months each)

started many years ago but
helped by more recent
classification results of
Nicolas & Ivan

(see that part of p. 5) So Hopf algebras that are graded are all simple subcoalgebras are one dimensional

come back to

Andruskiewitsch - Angiono - Petrova - W joint work IN PROGRESS

Prove conjecture for f.d. pointed Hopf algs / label gr of g-like elts

Let A be a f.d. pointed Hopf algebra with abelian group of g-like elts G

G -fin gr.

A Yetter-Drinfeld hG -module is a hG -module that is also a

(i.e. hG -comodule or hG -module)

G -graded module: $V = \bigoplus_{g \in G} V_g$ s.t. $h \cdot V_g \subset V_{hg^{-1}}$

$T(V)$ is a braided Hopf algebra, i.e. a Hopf algebra in the category ${}^{hG} \mathcal{YD}$

Fact There is a largest ideal J of $T(V)$, $J \subset \bigoplus_{n \geq 2} T^n(V)$,

that is also a coideal, i.e. $\Delta(J) \subset J \otimes T(V) + T(V) \otimes J$

Defn Nichols algebra $T(V)/J$

(there are more general and completely different definitions)

Ex $u_q(\mathfrak{sl}_2)^+ = k\langle E \mid E^n = 0 \rangle$ is a Nichols algebra ($q^n = 1$)

$u_q(\mathfrak{sl}_2)^{\neq 0} = k\langle E, K \mid E^n = 0, K^n = 1, KE = q^2 EK \rangle \cong u_q(\mathfrak{sl}_2)^+ \# k\langle K \rangle$

($u_q(\mathfrak{sl}_2)$ is quotient of Drinfeld double, deformed by a cocycle) is its bosonization

More generally: f.d. pointed Hopf algebras with abelian groups of g-like elts are cocycle deformations of

bosonizations of Nichols algebras (of diagonal type)

So go backwards to get: Outline of proof of conjecture for these std Hopf algs (of diagonal type) (chan o)

(1) Prove fg. of coh. of the Nichols algebra (Anick no, s.r., assoc, graded etc)

(2) Bosonization

(3) cocycle deformations via Drinfeld double results

(take advantage of D.d. construction, recent work of others)

$$\left(A fgc \leftarrow D(A) fgc \leftarrow \begin{matrix} gr A, (gr A)^* fgc \\ D(gr A) fgc \end{matrix} \right)$$

Angiono, Heckenberger

Nichols algebras are Hopf algebras in \mathcal{YD}

* NONABEL. GPS ARE THE WILD WEST classically not f.d.

Mention Dinkin diagrams classification in Nichols (of std)