SARAH WITHERSPOON: HOPF ALGEBRAS, II

Today, we will spend some time discussing non-semisimple Hopf algebras and tensor categories. This makes the classification question more complicated; there can be algebras or categories of wild type, where classifying all modules or objects, even the indecomposables, is just unrealistic.

So what can you do, then? It's still possible to make coarser classifications of objects and use techniques to gain partial information. Cohomology is particularly useful.

Definition 0.1. Let A be a Hopf algebra and $n > 0$. An *n-extension* of A-modules U and V is an exact sequence of A-modules

$$0 \longrightarrow V \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_2 \longrightarrow M_1 \longrightarrow U \longrightarrow 0. \tag{0.2}$$

A morphism of n-extensions is a commutative diagram

$$\begin{array}{ccccccccccccccc} 0 & \longrightarrow & V & \longrightarrow & M_n & \longrightarrow & \cdots & \longrightarrow & M_n & \longrightarrow & M_1 & \longrightarrow & U & \longrightarrow & 0 \\ & & \| & & \downarrow & & & & \downarrow & & \downarrow & & \| & & \\ 0 & \longrightarrow & V & \longrightarrow & M'_n & \longrightarrow & \cdots & \longrightarrow & M'_2 & \longrightarrow & M'_1 & \longrightarrow & U & \longrightarrow & 0, \end{array} \tag{0.3}$$

i.e. the maps on U and V are the identity. This does not define a symmetric relation on n-extensions, so define $\mathrm{Ext}^n_A(U, V)$ to be the set of n-extensions, modulo the smallest equivalence relation generated by morphisms.

There is an abelian group structure on $\mathrm{Ext}^n_A(U, V)$ induced by *Baer sum* of extensions.

Definition 0.4. The *Hopf algebra cohomology* of a Hopf algebra A over k is $H^n(A, k) := \mathrm{Ext}^n_A(k, k)$.

Hopf algebra cohomology carries a graded product structure.

Definition 0.5. Consider an m-extension and an n-extension of k by k, given respectively by

$$0 \longrightarrow k \longrightarrow M_n \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{\alpha} k \longrightarrow 0 \tag{0.6a}$$

$$0 \longrightarrow k \xrightarrow{\beta} N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow k \longrightarrow 0\ . \tag{0.6b}$$

The *Yoneda splice* of these two extensions is the $(m + n)$-extension

$$0 \longrightarrow k \longrightarrow M_m \longrightarrow \cdots \longrightarrow M_1 \xrightarrow{\beta\circ\alpha} N_n \longrightarrow \cdots \longrightarrow N_1 \longrightarrow k \longrightarrow 0. \tag{0.7}$$

Yoneda splice defines a bilinear map $H^m(A, k) \times H^n(A, k) \to H^{m+n}(A, k)$, called the *Yoneda product* or *cup product*; this makes $H^*(A, k) := \bigoplus_n H^n(A, k)$ into a graded ring.

We haven't used the Hopf algebra structure yet, and this cohomology ring exists for a general algebra.

Theorem 0.8. *If A is a bialgebra, then $H^*(A, k)$ is graded commutative.*

That is, this theorem uses comultiplication, but not the antipode.

More generally, given a tensor category $\mathcal{C}$, one can define a graded commutative k-algebra $H^*(\mathcal{C}, \mathbf{1})$.

Conjecture 0.9 (Friedlander-Suslin, Etingof-Ostrik)**.** If A is a finite-dimensional Hopf algebra, then $H^*(A, k)$ is finitely generated, and moreover, if U and V are finite-dimensional A-modules, $\mathrm{Ext}^*_A(U, V)$ is a finitely generated module over $H^*(A, k)$.[1]

Correspondingly, if $\mathcal{C}$ is a finite tensor category,[2] $H^*(\mathcal{C}, \mathbf{1})$ is finitely generated, and for any $X, Y \in \mathcal{C}$, $\mathrm{Ext}_{\mathcal{C}}(X, Y)$ is a finitely generated $H^*(\mathcal{C}, \mathbf{1})$-module.

Somehow this conjecture needs the fact that there *is* a comultiplication, but doesn't need the specific comultiplication, which is a little surprising.

[1] We haven't specified how to make this module structure; one way is to write $\mathrm{Ext}^*_A(U, V) \cong \mathrm{Ext}^*_A(k, U^* \otimes V)$ and form a Yoneda splice on the left.

[2] A *finite* tensor category is one satisfying a few niceness conditions, including that it has only finitely many simple objects.

Remark 0.10. There is another cohomology theory for algebras, called *Hochschild cohomology*. However, the analogue of Conjecture 0.9 for Hochschild cohomology is false! ◀

Why care about Conjecture 0.9? There is a theory of "varieties for modules" which is most useful in settings where the conjecture is true. The idea is to realize modules over noncommutative objects in terms of modules over commutative objects, and then take advantage of commutativity. Recent work of Bergh-Plavnik-Witherspoon [BPW19] works out a lot of this theory for general finite tensor categories.

Conjecture 0.9 is still open, but is known in a number of cases. Here are some established results.

- For $A = k[G]$ or $\mathcal{C} = \mathcal{R}ep_G$, G a finite group, this has been known for a long time. This is only interesting in modular characteristic (i.e. $\mathrm{char}(k) = p$ divides the order of G); otherwise, $k[G]$ is semisimple and its cohomology is concentrated in degree zero. This was established in the 1960s by Golodi, Venkov, and Evans; the theory of varieties for modules in this setting followed soon after.
- In positive characteristic, if A is a *restricted enveloping algebra*, i.e. a finite-dimensional quotient of $\mathcal{U}(\mathfrak{g})$, Conjecture 0.9 was established by Friedlander-Parshall [FP86, FP87].
- In characteristic zero, Conjecture 0.9 is true for the small quantum group $u_q(\mathfrak{g})$, as shown by Ginzburg-Kumar [GK93].
- In positive characteristic, if A is a finite-dimensional cocommutative Hopf algebra, Conjecture 0.9 was shwon by Friedlander-Suslin [FS97]. This was a significant breakthrough.

Some of these papers go beyond Conjecture 0.9, establishing structural results rather than just size.

The obstruction to understanding the general case is that we don't really understand finite-dimensional Hopf algebras and finite tensor categories well enough. But there has been recent progress, including work of Gordon (2000), Mastnak-Pevtsova-Schauenburg-Witherspoon [MPSW10], Bendel-Nukana-Parshall-Pillen (2014), Nguyen-Witherspoon [NW14] for twisted group algebras, Drupieski (2016) for supergroup schemes, Vay-Stefan (2016), Friedlander-Negron (2018) on Drinfeld doubles of cocommutative algebras, Nguyen-Wang-Witherspoon [NWW17, NWW19] in positive characteristic and a few general results; Erdmann-Silberg-Wang, Negron-Plavnik [NP18] recently on some general results on finite tensor categories; and more. There's been a lot of recent progress, but finishing off the conjecture will probably require new ideas.

Ongoing work of Andrurkiewitsch-Angimo-Pevstova-Witherspoon tackles the conjecture in characteristic zero for finite-dimensional pointed Hopf algebra whose grouplike elements form an abelian group — you always get a group, but the nonabelian case is wilder and a lot harder! This relies on previous results of Nicholas and Ivan on the structure theory of pointed Hopf algebras.

Yetter-Drinfeld modules are an important tool in the proof.

Definition 0.11. A *Yetter-Drinfeld* $k[G]$*-module* is a $k[G]$-module V together with a G-grading $V = \bigoplus_{g\in G} V_g$, such that for all $g, h \in G$, $h \cdot V_g = V_{hgh^{-1}}$. The category of Yetter-Drinfeld $k[G]$-modules is denoted ${}^{k[G]}_{k[G]}\mathcal{YD}$.

In the finite-dimensional case, these are equivalent to modules over the Drinfeld double of G.

Given a Yetter-Drinfeld module V, its tensor algebra $T(V)$ is a *braided Hopf algebra*, i.e. a Hopf algebra object in ${}^{k[G]}_{k[G]}\mathcal{YD}$. There is a largest ideal $J \subset T(V)$ that is also a *coideal*, i.e.

$$\Delta(J) = J \otimes T(V) + T(V) \otimes J. \tag{0.12}$$

This ideal J is concentrated in degrees greater than 1.

Definition 0.13. The *Nichols algebra* is $T(V)/J$.

Example 0.14. If q is an n^{th} root of unity, then $u_q(\mathfrak{sl}_2)^+ := k\langle E \mid E^n = 0\rangle$ is a Nichols algebra, and we can obtain

$$u_q(\mathfrak{sl}_2)^{\geq 0} = \langle E, K \mid E^n = 0, K^n = 1, KE = q^2 EK\rangle \tag{0.15}$$

as a smash product $u_q(\mathfrak{sl}_2)^+ \# k\langle K\rangle$; in this setting, the smash product is also called the *bosonization* of $u_q(\mathfrak{sl}_2)^+$. ◀

More generally, finite-dimensional pointed Hopf algebras whose group of grouplike elements are abelian arise not necessarily as bosonizations of Nichols algebras, but aren't far off; they're what's called *cocycle deformations*. This uses the classification of Nichols algebras, in terms of Dynkin and related diagrams.

References

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