ERIC ROWELL: AN INTRODUCTION TO MODULAR TENSOR CATEGORIES, II

Last time, we discussed a few different kinds of tensor categories, in particular pointed ribbon fusion categories and pointed modular tensor categories. Both of these have been classified; the classification amounts to finding compatible twists on $\mathcal{V}ec_G$ with various braidings.

Theorem 0.1 ([EGNO15])**.**

(1) *Pointed ribbon fusion categories up to equivalence are classified by data of a finite abelian group G and a quadratic form $q\colon G \times G \to \mathbb{C}^\times$.*
(2) *Pointed modular tensor categories are classified by (G, q) as above, subject to the condition that q is nondegenerate.*

The data of (G, q) is often called a *pre-metric group*, and if q is nondegenerate, it's called a *metric group*. The quadratic form determines the 2-cocycle that specified the braiding, via

$$B(g,h) \coloneqq \frac{q(g)(q(h)}{q(gh)}. \tag{0.2}$$

This is all very nice, but we would like some more interesting examples, so we turn to quantum groups $\mathcal{C}(\mathfrak{g}, \ell)$. Here $\mathfrak{g}$ is a simple Lie algebra and $\mathcal{C}$ is the category of modules over $\mathcal{U}_q(\mathfrak{g})$, where $q \coloneqq \exp(\pi i/m \ll)$. For $m = 1$, $\mathfrak{g}$ can be ADE type; for $m = 2$, of BCF type; and for $m = 3$, $\mathfrak{g} = \mathfrak{g}_2$. Setting up the category involves some technical details, but can be done, and we obtain modular categories![1]

Example 0.3. Let's take $\mathfrak{g} = \mathfrak{so}_5$ and $\ell = 5$, so $q = e^{i\pi/10}$. The objects in $\mathcal{C}$ are described by a Weyl chamber for $\mathfrak{g}$, but $\ell = 5$ imposes that we kill all objects above a certain line. In this we have the standard representation V, the adjoint representation A, and an object at coordinates $(1/2, 1/2)$ with quantum dimension $\sqrt{5}$. The level (in the notation of the previous talk) of this category is 2, so sometimes it's also denoted $\mathrm{SO}(5)_2$. ◂

Example 0.4. Let's consider $\mathcal{C}(\mathfrak{sl}_2, 5)$. Now we look at a ray within the one-dimensional root space, and only keep the first three objects, S at 1, A at τ, and the unit. The fusion rules are $A^{\otimes 2} = \mathbf{1} \oplus A$, and $S^{\otimes 2} \cong \mathbf{1}$. Thus this category actually splits as a Deligne tensor product of the subcategory generated by S, which is called the *semion category*, and the subcategory generated by A, which is called the *Fibonacci category*. Both of these are fundamental examples. ◂

Example 0.5. $\mathcal{C}(\mathfrak{sl}_2, 4)$ is an *Ising category*. Its simple objects are $\mathbf{1}$, σ, and ψ. Here $\dim(\sigma) = \sqrt{2}$, $\dim(\psi) = 1$, $\theta_\sigma = e^{3\pi i/8}$, and $\theta_\psi = -1$. This σ particle was the first nonabelian anyon discovered, and it's reminiscent (though not the same as) to a Majorana fermion. The S-matrix is

$$S = \begin{pmatrix} 1 & \sqrt{2} & 1 \\ \sqrt{2} & 0 & -\sqrt{2} \\ 1 & -\sqrt{2} & 1 \end{pmatrix}. \tag{0.6}$$ ◂

We've described examples of modular categories via their *modular data*: the S-matrix and also the T*-matrix* $T_{ij} = \delta_{ij}\theta_i$. Stay tuned for a talk later this weey by Colleen Delaney with more details.

The modular group $\mathrm{SL}_2(\mathbb{Z})$ is generated by two matrices $S = \left(\begin{smallmatrix} 0 & -1 \\ 1 & 0 \end{smallmatrix}\right)$ and $T = \left(\begin{smallmatrix} 1 & 1 \\ 0 & 1 \end{smallmatrix}\right)$. The S- and T-matrices appearing in the data of a modular category satisfy relations that imply they define a projective representation Φ of $\mathrm{SL}_2(\mathbb{Z})$.

Theorem 0.7 (Ng-Schauenburg [NS10])**.** *The image of such a representation Φ is finite. In fact, if N is the order of T, then Φ factors over* $\mathrm{SL}_2(\mathbb{Z}/n)$.

Classifying fusion categories is too difficult in general, but modular categories have more adjectives in front of them. Maybe we can classify them, at least for a fixed rank r that's not too large. Or even, how many of them are there?

[1]Here m is important; if you leave it out, you'll always get a ribbon category, but not necessarily a modular one.

A good first step is to consider the field $\mathbb{K}_0 := \mathbb{Q}(s_{ij})$, which sits inside $\mathbb{Q}(\theta_i)$. Since T has finite order, $\mathbb{Q}(\theta_i)$ is a cyclotomic extension $\mathbb{Q}(\zeta_N)$ for some primitive N^{th} root of unity ζ_N. These are particularly nice Galois extensions in that:

(1) Since $\mathbb{Q} \hookrightarrow \mathbb{Q}(\theta_i)$ is a cyclotomic extension, $\mathrm{Gal}(\mathbb{Q}(\theta_i)/\mathbb{Q})$ is abelian, and in particular always solvable.
(2) Since we're looking at rank r, the T-matrix is $r \times r$, so we get an embedding $\mathrm{Gal}(\mathbb{K}_0/\mathbb{Q}) \hookrightarrow \mathrm{Aut}(\mathrm{Irr}(\mathcal{C})) \cong S_r$.
(3) There is some k such that $\mathrm{Gal}(\mathbb{Q}(\zeta_N)/\mathbb{K}_0) \cong (\mathbb{Z}/2)^k$.

Thus we have a recipe for classifying modular categories of rank r.

(1) Choose an abelian subgroup A of S_r. Then, using the above facts, classify all possible S-matrices which yield the Galois group $\mathrm{Gal}(\mathbb{Q}(\mathbb{K}_0/\mathbb{Q}) \cong A \subset S_r$. For many choices of A, there are no possible S-matrices.
(2) The *Verlinde formula* determines the fusion rules from the S-matrix.
(3) Finally, an analogue of Ocneanu rigidity (**??**) informs us that there are finitely many modular tensor categories with fixed fusion rules.

This has worked completely up to rank 5 so far, and is also effective in rank 6. One general question, which is still open, is *if you fix a fusion category, how do you classify its possible modular structures?* We know there can only be finitely many, but that theorem is nonconstructive. In special cases, things are known; for example, a result of Kazhdan-Wenzl [KW93] allows us to solve this for $\mathcal{C}(\mathfrak{sl}_n, \ell)$. More recent work of Nikshych [Nik19] establishes how to classify the possible braidings given fixed fusion rules. And spherical structures on a modular tensor categories are understood: they're given by invertible objects with order at most 2.

Theorem 0.8 (Rank-finiteness (Bruillard-Ng-Rowell-Wang [BNRW16]))**.** *There are finitely many modular tensor categories of a fixed rank r.*

The proof ultimately relies on results in analytic number theory, which is interesting.

Moving on, let $\mathcal{C}$ be a braided fusion category and B_n denote the braid group on n strands. Given an object $X \in \mathcal{C}$, the braiding defines a map $\psi\colon B_n \to \mathrm{Aut}(X^{\otimes n})$; if σ_i denotes the braid that switches braids i and $i+1$, then

$$\psi(\sigma_i) := \mathrm{id}_X^{\otimes(i-1)} \otimes c_{X,X} \otimes \mathrm{id}_X^{(n-i-1)}. \tag{0.9}$$

$\mathrm{Aut}(X^{\otimes n})$ acts on

$$\mathcal{H}_n^X := \bigoplus_{Y \in \mathrm{Irr}(\mathcal{C})} \mathrm{Hom}(Y, X^{\otimes n}), \tag{0.10}$$

so we get a representation $\rho_X\colon B_n \to \mathrm{GL}(\mathcal{H}_n^X)$. In addition to being an interesting braid group representation on its own, this representation is important for implementing gates in topological quantum computation.

It's natural to ask whether the image of ρ_X is finite.

Definition 0.11. We say that $X \in \mathcal{C}$ has *property F* if the image of ρ_X is finite.

The Ising category (or rather, its nontrivial simple object) has property F, but the Fibonacci category does not.

Definition 0.12. Let X be an object in a fusion category $\mathcal{C}$ and N_X be the matrix of fusion with X on $\mathrm{Irr}(\mathcal{C})$, i.e.

$$(N_X)_{ij} = \dim \mathrm{Hom}_{\mathcal{C}}(X \otimes X_j, X_i). \tag{0.13}$$

The *Frobenius-Perron dimension* of X, denoted $\mathrm{FPdim}(X)$, is the largest eigenvalue of N_X. If X is simple and $\mathrm{FPdim}(X)^2 \in \mathbb{Z}$, X is called *weakly integral.*

Over 10 years ago, the speaker conjectured that X is weakly integral iff it has property F. This is known in special cases.

- For pointed fusion categories, this is essentially an exercise.
- For group-theoretical braided fusion categories (e.g. $\mathcal{R}ep(D^\omega G)$), this is due to Etingof-Rowell-Witherspoon [ERW08].

- For quantum groups $\mathcal{C}(\mathfrak{g}, \ell)$, this is known, thanks to work of Jones, Freedman, Larsen, Wang, Rowell, and Wenzl.
- Recently, this conjecture has been verified for weakly group-theoretical braided fusion categories by Green-Nikshych [GN19]. There is a different conjecture that weakly group-theoretical is equivalent to weakly integral.

This veracity of this conjecture is closed under taking Deligne tensor products, Drinfeld doubles, and a few other useful operations.

There are still many interesting open questions! For example, from a nondegenerate braided fusion category, one canextract an invariant called the *Witt group*, and this seems to be a rich and interesting invariant that we are still in the process of understanding.

References

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