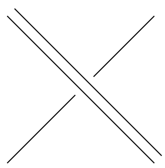
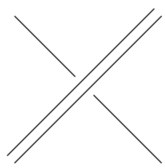
ERIC ROWELL: AN INTRODUCTION TO MODULAR TENSOR CATEGORIES, I

In this lecture, we'll begin with definitions and basic examples of modular tensor categories, and then use them in the next lecture. But first, let's discuss the whys of modular tensor categories.

We're often interested in knot and link invariants which are pictorial in nature, e.g. computed using a diagram. Another seemingly unrelated application is to study statistical-mechanical systems. Witten introduced TQFT into this story, extending the Jones polynomial to 3-manifold invariants using physics. Lately, there are interesting condensed-matter pheomena in topological phases. All of these are goverened by modular tensor categories in different ways, and in related ones.

Definition 0.1. Let $\mathcal{C}$ be a fusion category. A *braiding* on $\mathcal{C}$ (after which it's called a *braided fusion category*) is data of a natural transformation $c_{X,Y} \colon X \otimes Y \xrightarrow{\cong} Y \otimes X$ satisfying some relations called the *hexagon identities*.

You can think of $c_{X,Y}$ as taking strands labeled by the objects X and Y, and laying the X strand over the Y strand. The hexagon identities arise by comparing the two strands

(0.2)  and  .

Because the braiding is implemented via a natural transformation, it is functorial: we can braid morphisms as well as objects.

Example 0.3. Given a finite group G, $\mathcal{R}ep_G$ is a braided fusion category. Let V and W be representations; then the braiding $c_{V,W}(v \otimes w) := w \otimes v$. ◀

Definition 0.4. Let $\mathcal{C}$ be a braided fusion category. The *symmetric center* or *Müger center* of $\mathcal{C}$ is the subcategory $\mathcal{C}'$ of $x \in \mathcal{C}$ such that $c_{X,Y} c_{Y,X} = \mathrm{id}_X$ for all $Y \in \mathcal{C}$.

For example, the symmetric center of $\mathcal{R}ep_G$ is once again $\mathcal{R}ep_G$.

Exercise 0.5. Why is the symmetric center of $\mathcal{C}$ a braided fusion category? In particular, why is it closed under tensor products?

Definition 0.6. If the symmetric center of $\mathcal{C}$ is itself, we call $\mathcal{C}$ *symmetric*.[1] If the symmetric center of $\mathcal{C}$ is generated by the unit object (equivalently, $\mathcal{C}' \simeq \mathcal{V}ect$), we call $\mathcal{C}$ *nondegenerate*.

Here, "generated by the unit object" means every object is isomorphic to a direct sum of copies of the unit.

Now let's put some more adjectives in front of these structures. These will make the structure nicer, as usual, but are interesting enough to have examples.

Definition 0.7. Let $\mathcal{C}$ be a braided fusion category. A *twist* on $\mathcal{C}$ is a choice of $\theta \in \mathrm{Aut}(\mathrm{id}_{\mathcal{C}})$.

Diagrammatically, we think of the twist as acting by the diagram in the first Reidemeister move, except we place right over left, not left over right. By looking at a picture of the twist on $X \otimes Y$, and untangling the picture, you can prove the *balancing equation*

$$\theta_{X \otimes Y} = c_{X,Y} \circ \theta_X \otimes \theta_Y. \tag{0.8}$$

Diagrams make it easier to picture these relations, but aren't strictly necessary. For example, the evaluation map $d_X \colon X^* \otimes X \to \mathbf{1}$ is represented by a diagram $\frown$ labeled by X, and coevaluation $b_X \colon \mathbf{1} \to X^* \otimes X$ is represented by a diagram $\smile$ labeled by X. Since braided categories aren't necessarily symmetric, one must be careful with left versus right duals.

[1] Notice that being symmetric is a property of braided fusion categories.

Definition 0.9. A *ribbon structure* on a braided fusion category $\mathcal{C}$ is a twist such that $(\theta_X)^* = \theta_{X^*}$.

Here's where it's useful to use ribbon diagrams rather than string diagrams: really we want to keep track of the normal framings of the strings in our diagrams (thought of as embedded in $\mathbb{R}^3$), and ribbons provide a clean way to understand that.

Let $\mathrm{Irr}(\mathcal{C})$ denote the set of isomorphism classes of irreducible objects in $\mathcal{C}$. This is always a finite set; the *rank* of $\mathcal{C}$ is $\#\mathrm{Irr}(\mathcal{C})$. Choose representatives $x_1, \dots, x_r$ of the isomorphism classes of simple objects; then, by Schur's lemma, $\mathrm{Aut}(X_i) \cong \mathbb{C}^\times$. Let $\theta_i \in \mathbb{C}^\times$ denote the twist of X_i.

Now we have all the words we need to define modular tensor categories.

Definition 0.10. A *modular tensor category* is a nondegenerate ribbon fusion category.

There are other, equivalent definitions.

Definition 0.11. A *pivotal structure* on a fusion category $\mathcal{C}$ is a natural isomorphism $j\colon X \overset{\cong}{\to} X^{**}$.

If a pivotal structure satisfies a certain niceness condition, it's called *spherical.* Then:

- A braided fusion category with a pivotal structure automatically has a twist.
- If that pivotal structure is spherical, the twist defines a ribbon structure.
- A nondegenerate braided fusion category with a spherical structure is a modular tensor category.

This still hasn't quite made contact with the usual definition.

If $\mathcal{C}$ is a ribbon fusion category, it has a canonical trace on $\mathrm{End}(X)$, valued in $\mathrm{End}(\mathbf{1}) \cong \mathbb{C}$. The *dimension* of an object $X \in \mathcal{C}$ is $\mathrm{tr}(\mathrm{id}_X)$.

Definition 0.12. The *S-matrix* of a ribbon fusion category is the matrix with entries $S_{ij} \coloneqq \mathrm{tr}(c_{X_i,X_j} \circ c_{X_j,X_i})$ for $X_i, X_j \in \mathrm{Irr}(\mathcal{C})$.

Theorem 0.13 (Brugières-Müger)**.** *A ribbon tensor category $\mathcal{C}$ is modular if and only if the S-matrix is invertible.*

Now let's turn to examples.

Example 0.14. Let G be a finite abelian group and $\mathcal{V}ec_G$ be the category of G-graded vector spaces. These were discussed previously in **??**, albeit in a slightly different way.

Let $c\colon G \times G \to \mathbb{C}^\times$ be a *bicharacter* of G, i.e. for all $g, h, k \in G$,

$$c(gh, k) = c(g, k)c(h, k). \tag{0.15}$$

Then we obtain a braiding on $\mathcal{V}ec_G$ by $c\colon g \otimes h \to h \otimes g$ by

$$\theta_g(v \otimes w) = c(g, h) w \otimes v. \tag{0.16}$$

For the twist, use $\theta_g \coloneqq c(g, g)$. This defines a ribbon tensor category, and it is modular iff $\det((c(g,h)c(h,g))_{g,h}) \neq 0$.

Exercise 0.17. In particular, let $G \coloneqq \mathbb{Z}/3$ and w be a generator. Show that $c(w, w) = \exp(2\pi i/3)$ extends to a bicharacter that defines a modular tensor structure on $\mathcal{C} \coloneqq \mathcal{V}ec_G$. Show that we cannot obtain a modular structure on $\mathcal{V}ec_{\mathbb{Z}/2}$ in this way, however.

We can produce a modular structure on $\mathcal{V}ec_{\mathbb{Z}/2}$ in a different way: let z be a generator, and define $c(z, z) \coloneqq i$ and $c(1, z) = c(z, 1) = c(1, 1) = 1$. This defines a modular tensor category structure on $\mathcal{V}ec_{\mathbb{Z}/2}^\omega$ whenever ω is cohomologically nontrivial; this category is of considerable interest in physics, where it's known as the *semion category.* ◀

If you tried to generalize this to G nonabelian, you would not be able to write down a braiding, because $g \otimes h \ncong h \otimes g$.

If all simple objects in $\mathcal{C}$ are invertible, $\mathcal{C}$ is called a *pointed fusion category.* It turns out these have been classified, and the underlying monoidal tensor category id $\mathcal{V}ec_G^\omega$ for some finite group G and some cocycle ω. If in addition $\mathcal{C}$ is braided, then G is abelian, and we can ask about the converse.

Theorem 0.18. *If $|G|$ is odd, $\mathcal{V}ec_G^\omega$ admits a braiding iff ω is cohomologically trivial.*

When $|G|$ is even, things are more complicated, as we saw above, but the answers are known. For $\mathbb{Z}/2$, we can get $\mathcal{R}ep_{\mathbb{Z}/2}$, and for $c(z, z) = -1$, we obtain $s\mathcal{V}ec$. Both of these are symmetric. One can generalize: Deligne [Del02] classified symmetric fusion categories, showing they're all equivalent to $\mathcal{R}ep_G$ or $\mathcal{R}ep_G(z)$, where $z \in G$ is central and order 2 (giving a super-vector space structure on G-representations). Symmetric fusion categories equivalent to $\mathcal{R}ep_G$ are called *Tannakian*; those equivalent to $\mathcal{R}ep_G(z)$ are called *super-Tannakian*.

References

[Del02] P. Deligne. Catègories tensorielles. volume 2, pages 227–248. 2002. Dedicated to Yuri I. Manin on the occasion of his 65th birthday. 3