ANNA BELIAKOVA: QUANTUM INVARIANTS OF LINKS AND 3-MANIFOLDS, II

Recall that we're in the business of studying non-semisimple quantum invariants of knots and 3-manifolds. Last time, we discussed how surgery on a framed link can turn logarithmic framed link invariants into 3-manifold invariants, as studied by Hennings (1998), Beliakova-Blanchet-Geer [BBG18], and Costantino-Geer-Patureau-Mirand [CGPM14].

First, the algebraic data that we need. Let H be a finite-dimensional Hopf algebra, such as

$$u_\xi \coloneqq u_q(\mathfrak{sl}_2) \otimes_{\mathcal{A}} \mathbb{C}, \tag{0.1}$$

where $\mathcal{A} \coloneqq \mathbb{Z}[q^{\pm 1}]$, ξ is a p^{th} root of unity, and $\mathcal{A}$ acts on $u_q(\mathfrak{sl}_2)$ by $q \mapsto \xi$. We can also restrict to $u^{rest} \coloneqq u_\xi / \langle e^p, F^p, K^{2p} - 1\rangle$; then K^p is central.

Radford showed that there exists a unique $\mu^* \in H^\times$, called the *integral*, such that

$$(\mu \otimes \mathrm{id})\Delta(x) = \mu(x)\mathbf{1} \tag{0.2}$$

for all $x \in H$. For example, in u_ξ,

$$\mu(E^m F^n K^\ell) = \delta_{m,p-1}\delta_{n,p-1}\delta_{\ell,p+1}. \tag{0.3}$$

Theorem 0.4 (Hennings, Kauffman-Radford)**.** *Let L be a framed link and $M \coloneqq S^3(L)$. Let σ_+, resp. σ_-, be the number of positive, resp. negative eigenvalues of $\ell k(L)$. Then the Hennings invariant can be calculated as*

$$\mathrm{Hen}(M) = \frac{\mu^{\otimes |L|}(J_L)}{\mu(J_{U_+})^{\sigma_+}\mu(J_{u_-})^{\sigma_-}}. \tag{0.5}$$

The proof is surprisingly simple.

Proof. Using the Kirby move K1, write $L' = L \amalg u_+$, where u_+ indicates an unlinked unknot. Then

$$\mathrm{Hen}(M) = \frac{\mu^{\otimes |L|}(J_L)\mu(J_{u_+})}{\mu(J_{u_+}^{\sigma_+ + 1})\mu(J_{u_-})^{\sigma_-}}, \tag{0.6}$$

and we cancel out the factors of $\mu(J_{u_+})$ in the numerator and denominator. Now, perform the Kirby move K2 and use (0.2) and we're done. ⊠

Remark 0.7. If H is semisimple,

$$\mu = \sum \mathrm{qdim}(V_i)\, \mathrm{tr}_q^{V_i}, \tag{0.8}$$

where the sum is over the isomorphism classes of irreducible modules V_i, and the Hennings invariant for H and M and the Witten-Reshetikhin-Turaev invariant for H and M coincide. ◀

Kuperberg [Kup91] constructed a related invariant using cointegrals in a Hopf algebra.

Definition 0.9. Let H be a Hopf algebra. A *left cointegral* is an element $c \in H$ satisfying $xc = \varepsilon(x)c$ for all $x \in H$; a *right cointegral* satisfies $cx = \varepsilon(x)c$ for all $x \in H$.

A Hopf algebra in which left and right cointegrals coincide is called *unimodular*.

Kuperberg's invariant is the simplest algebraic 3-manifold invariant one can define with Hopf algebras, and this lends it its usefulness — it will probably be one of the first things we fully categorify. For example, if we chose $u_q(\mathfrak{sl}_2)$, we'd need the data of the Borel part, $\langle K, E\rangle$. Chang-Cui [CC19] showed that the Kuperberg invariant for M and H coincides with the Hennings invariant for M and $\mathcal{D}(H)$, the double of H, analogous to the relationship between Reshetikhin-Turaev invariants for a modular tensor category and Turaev-Viro invariants for its Drinfeld double.

Theorem 0.10 (Chen-Kuppum-Srinivasan [CKS09])**.** *If $b_1(M) = 0$, the Hennings invariant of M is $|H_1(M)|\, WRT(M)$; otherwise, it vanishes.*

Proof. Our proof sketch follows Habiro-Lê. Note: I (the notetaker) didn't follow what was written on the board; sorry about that. I think what happened was: the Hennings invariant of M ends up being $\sum_{i\in I} x_i \otimes y_i$, where $\{x_i\}$ and $\{y_i\}$ are both bases of H. This induces a Hopf pairing, sometimes called the *quantum Killing form*, by declaring $\langle x_i, y_j\rangle \coloneqq \delta_{ij}$. Hence if $x = \sum a_i y_i$, $\langle x, x_i\rangle = a_i$.

Let M be an integral homology sphere, so we can realize M as surgery on a knot K framed with framing ± 1. Then $I_M = \langle r^{-1}, J_K\rangle$, where r is a *ribbon element* in H. We claim that for all $x \in H$,

$$\langle r^{-1}, x\rangle = \frac{\mu(x^r)}{\mu(r)}, \tag{0.11}$$

and that $\Delta(r) = (r \otimes r)M$. This is because $(\mu \otimes \mathrm{id})\Delta(r) = \mu(r)\mathbf{1}$, so

$$\sum_i \mu(rx_i)y_i = \mu(r)\mathbf{1}, \tag{0.12}$$

and therefore

$$r^{-1} = \sum_i \frac{\mu(rx_i)}{\mu(r)} y_i \tag{0.13}$$

$$\langle r^{-1}, x_i\rangle = \frac{\mu(rx_i)}{\mu(r)}. \tag{0.14}$$

If $b_1(M) > 0$, $S^2 \times S^1 = S^3(U_0)$, where U_0 denotes an unknot, and $J_{U_0} = \mathbf{1}$, so the Hennings invariant is $\mu(\mathbf{1}) = 0$. ⊠

This seems to spell doom for non-semisimple invariants, but not all of them are killed. This leads one to introduce *modified traces*, following Geer-Patureau (2008), functions $t_P \colon \operatorname{End} P \to k$, where P is an H-module, such that $t_P(fg) = t_P(gf)$ and t_P satisfies the *partial trace property* in Figure 1.

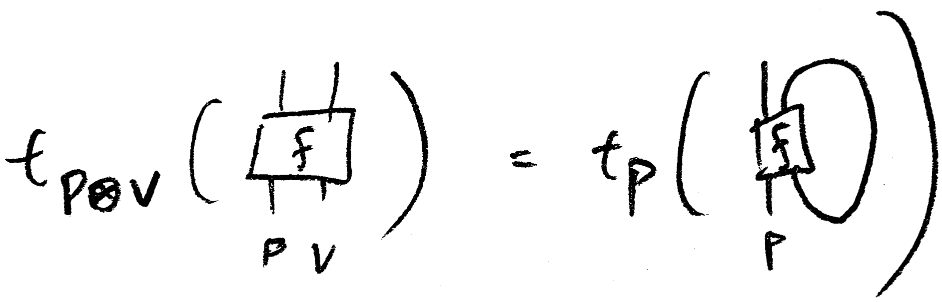

Figure 1. The partial trace property.

Remark 0.15. Now taking the invariant J_K as usual, this kind of invariant lands in

$$qHH_0(H) \coloneqq H/\langle xy0S^2(y)x \mid x, y \in H\rangle. \tag{0.16}$$

◀

Theorem 0.17 (Beliakova-Blanchet-Gainutdinov [BBG17])**.** *Let H be a unimodular pivotal Hopf algebra. Then for $f \in \operatorname{End}_H(H)$,*

$$\operatorname{tr}_H(f) = \mu_g(f(1)), \tag{0.18}$$

where g is the pivotal element: $\mu_g(x) = \mu(gx)$. In particular, the modified trace is uniquely determined.

Then, Beliakova-Blanchet-Geer [BBG18] used this to define more invariants for a knot K_P in a 3-manifold $M = S^3(L)$: the invariant is $(\mu^{\otimes|L|} \otimes t_P)J_{L\cup K_P}$. These invariants were extended to a TQFT by De Renzi, Geer, and Patureau-Mirand [DRGPM18].

In the last few minutes, we'll discuss CGP invariants. Consider the Hopf algebra

$$u^{\text{unrolled}} \coloneqq \langle K, E, F, H\rangle/\langle E^p, F^p\rangle. \tag{0.19}$$

Given $\lambda \in \mathbb{C}$, we have a p-dimensional irreducible u^{unrolled}-module V_λ, though thanks to some redundancy, really the classification is in $\mathbb{C}/2\mathbb{Z}$. This leads to an invariant of a manifold together with a cohomology class $\lambda \in H^n(M; \mathbb{C}/2\mathbb{Z})$, given by

$$\mathrm{CGP}(M, \lambda) \coloneqq \sum_{k=0}^{p-1} d^{mod}(V_{\lambda+2k}) J_K(V_{\lambda+2k}) \tag{0.20}$$

This is a surprisingly simple description of this kind of invariant, which is nice. At $p = 2$, this specializes to the Alexander polynomial and Reidemeister torsion. Blanchet, Costantino, Geer, and Patureau-Mirand [BCGPM16] extended this invariant to a TQFT in which the order of the Dehn twist is trivial. The S- and T-matrices of this TQFT are related to work of Gukov and collaborators.

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MSRI Lecture 2 -1-

Quantum non-semisimple invariants of links and 3-mfs II

$$\begin{tikzcd}
\left\{\begin{matrix}\text{framed} \\ \text{links in} \\ S^3\end{matrix}\right\} \arrow[r, "\text{surgery}"] \arrow[d, "\mathcal{U}^{rest}\ \text{f.d},\ \mathcal{U}^{unrolled}"'] & \left\{\begin{matrix}\text{cpt oriented} \\ \text{3-mfs}\end{matrix}\right\} \arrow[d] \\
\left\{\begin{matrix}\text{logarithmic} \\ \text{invariants}\end{matrix}\right\} \arrow[r, "\text{“integration”}"] & \left\{\begin{matrix}\text{Hennings '98} \\ \text{B-Blanchet-Geer '17} \\ \text{Costantino-Geer-Patureau (CGP)}\end{matrix}\right\}
\end{tikzcd}$$

① Algebraic data for Hennings and BBG

H f. dim ribbon Hopf algebra

Main example: $\xi^p = 1$, $\mathcal{U}^{rest} = \langle e, F^{(n)}, K\rangle_{q=\xi} \big/ \langle e^p, F^p, K^{2p}-1\rangle$

$e = \{1\}E = (v - v^{-1})E$

Later: $Kv = \lambda v \overset{K^{2p}=1}{\Rightarrow} \lambda^{2p} = 1 \Rightarrow \lambda$ is $2p$-th root of unity

weight vector "all modules have integral weights"

$K^p \in Z(\mathcal{U}^{rest})$ idempotent ~~invertible~~, $K^p = \pm 1$ "small" is where K^p acts by $+1$

Book Radford on Hopf algebras

$\exists! \ \mu \in H^* \qquad (\mu \otimes id)\Delta(x) = \mu(x)1 \quad \forall x \in H$

Example: $\mu(E^m F^n K^l) = \delta_{m,p-1}\,\delta_{n,p-1}\,\delta_{l,p+1}$

Thm: (Hennings, Kauffman-Radford) Let $M = S^3(L)$

$|L|$ # link compon

$$Hen(M) = \frac{\mu^{\otimes |L|}(J_L)}{\mu(J_{U_+})^{\sigma_+}\,\mu(J_{U_-})^{\sigma_-}}$$

topolog. invariant of M

$\sigma_\pm$ # $\substack{\text{pos.}\\ \text{neg.}}$ eigenvalues of $lk(L)$.

Proof: K1: $L' = L \sqcup U_+$

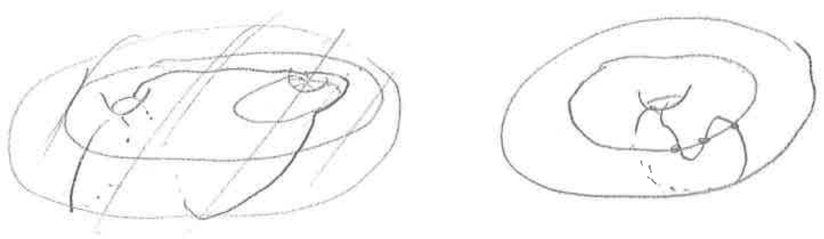
$$\frac{\mu^{\otimes |L|}(J_L)\,\cancel{\mu(U_+)}}{\mu(J_{U_+})^{\sigma_+ + 1}\,\mu(J_{U_-})^{\sigma_-}}$$

K2:

$\mu(x)1 \quad = \quad (\mu \otimes id)\Delta(x)$ ∎

<u>Remark</u>: $\exists$ another construction due to Kuperberg '98 using integral + cointegral $c \in H$

s.t. $x \cdot c = \varepsilon(x) c$ left H is unimodular if left integ is right.
$c \cdot x = \varepsilon(x) c$ right



involutive $S^2 = id$

<u>Advantage</u>: minimal algebraic data, no braiding and ribbon structure requir.

Borel subalgebra works.

$$\mathrm{Kup}(M, \overset{b}{H}) = \mathrm{Hen}(M, \overset{\phi(b)}{\mathcal{D}(H)})$$

Chang - Cui '17

If H is semisimple (Kerler) '95 $\mu = \sum q\dim V_i \, tr_q^{V_i}$

$$WRT(M) = \mathrm{Hen}(M)$$

$$\mathrm{Kup}(M) = \mathrm{Hen}(M \# -M)$$

$\Rightarrow$ Hen(M) looks like a non-semisim. generalization, however

<u>Thm</u> (Chen - Kuppum - Srinivasan) '07

$$\mathrm{Hen}(M) = \begin{cases} |H_1(M)| \, WRT(M) & \text{if } b_1(M) = 0 \\ 0 & b_1(M) > 0 \end{cases}$$

<u>Prof</u> (Habiro-Le): U^{rest} is factorizable

$\Rightarrow \quad M = R_{21} R = \sum x_i \otimes y_i$ $\quad \{x_i\}, \{y_i\}$ are base of U^{rest}

Hopf pairing $\quad \langle x_i, y_j \rangle = \delta_{ij}$
(quantum Killing form)

$\Rightarrow \quad x = \sum a_i y_i = \sum \langle x, x_j \rangle y_j$

Let M be a $\mathbb{Z}HS$
The unified Habiro's invariant of M

$$\boxed{I_M = \langle r', J_K \rangle}$$

Claim: $\quad \langle r', x \rangle = \dfrac{\mu(xr)}{\mu(r)} \quad \forall x \in H$

Indeed,
$r' \mid \quad (\mu \otimes id) \Delta r = \mu(r) \mathbb{1} = \sum_i \mu(r x_i) y_i$

$\Delta r = (r \otimes r) M$

$$r' = \sum_i \frac{\mu(r x_i)}{\mu(r)} y_i$$

$$\langle r', x_i \rangle = \frac{\mu(r x_i)}{\mu(r)}$$

② Case $b_1 > 0$

$$S^2 \times S^1 = S^3(U) \qquad J_{U} = 1 \quad \mu(J_{4_1}) = 0$$

Q: Does it kill the non-semisimple story?

A: No:

① Improvement. $\quad H = \oplus P_i$

 $= 0$

$\forall f \in \mathrm{End}\, P \in H\text{-pmod}$

$$P \oplus P^{\perp} = P_1$$

Solution: Modified trace

$$\{ t_P : \mathrm{End}\, P \to \mathbb{K} \}_{P \in H\text{-pmod}}$$

- $t_P(fg) = t_P(gf)$
- $t_{P\otimes V}\Big($  $\Big) = t_P\Big($  $\Big)$ $\quad V \in H\text{-mod}$

Remark:

$$\begin{tikzcd} \{\text{Knots}\} \arrow[r, "J_K"] \arrow[dr] & H \arrow[r, "tr_q^V"] \arrow[d, "tr^H"] & \mathbb{K} \\ & qHH_0(H) & \end{tikzcd}$$

categorical trace

$$q HH_0(H) = H \Big/ \{xy - yS^2(x) \mid x, y \in H\}$$

t factorizes through $HH_0(H)$, but is compatible with categorical trace

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② BBG invariant $\quad M = S^3(L)$

$$BBG(M, K_p) = (\mu^{\otimes |K|} \otimes t_p)(J_{L \cup K_p})$$

non-zero if $b_1 > 0$, if we add an unknot.

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① Thm (B-Blanchet-Gainutdinov)

$\forall$ unimodular pivotal H

$$t_H(f) = \mu_g(f(1)) \quad \forall f \in \mathrm{End}_H H$$

g pivotal element

$\mu_g(x) = \mu(gx)$ ← this form has a partial trace property suitable to define non-semisim invariants

<u>Corol:</u> ~~If~~ H-mod is semisimple (unimodular pivotal) $\Leftrightarrow$ its categorical trace $\neq 0$. then $tr^c = \mu_g$.

③ CGP $\quad \mathcal{U}^{\text{unrolled}} = \mathcal{U}_\xi / \langle e_1^p, F^p \rangle$

What happens? $\quad K v = \lambda v \qquad$ if $K^{2p} = 1 \Rightarrow \lambda^{2p} = 1$

here we have non-integ. weights

We have $\lambda \in \mathbb{C}$!

irrep proj. V_λ of dim p for $\lambda \in \mathbb{C}$

$\lambda \in H^1(M, \mathbb{C}/2\mathbb{Z}) \qquad t_{V_\lambda}(\mathrm{id}_{V_\lambda}) = d(V_\lambda)$ modified dimension

$$CGP(M, \lambda) := \sum_{k=0}^{p-1} d(V_{\lambda+2k}) \, J_k(V_{\lambda+2k})$$

(up to normalization)

Indices (D. Lopez categorified)

(0) at $p=2$ $\Delta_K(t)$, Reidemeister torsion $=$

Properties: ① extends to TQFT (BCGP, De Renzi-Geer-Patureau) $\begin{pmatrix} q^{\lambda} & 1 \\ 0 & q^{\lambda} \end{pmatrix}$

② the order of the twist is ∞

③ Gunov observe S, T matrices from non-semi TQFT in their construction

④ On going project with O. Blanchet

we are relating $J_k(V_{\lambda+2k})$ to a deformed cyclotomic expansion of Habiro

and hope to have a universal version of these Laplace transform to invariants

relate CGP with CWT C of the classical invariants