Recall that we’re in the business of studying non-semisimple quantum invariants of knots and 3-manifolds. Last time, we discussed how surgery on a framed link can turn logarithmic framed link invariants into 3-manifold invariants, as studied by Hennings (1998), Beliakova-Blanchet-Geer [BBG18], and Costantino-Geer-Patureau-Mirand [CGPM14].

First, the algebraic data that we need. Let $H$ be a finite-dimensional Hopf algebra, such as
\begin{equation}
(0.1) \quad u_\xi := u_\xi(\mathfrak{sl}_2) \otimes_A \mathbb{C},
\end{equation}
where $A := \mathbb{Z}[q^{\pm 1}]$, $\xi$ is a $p^{th}$ root of unity, and $A$ acts on $u_\xi(\mathfrak{sl}_2)$ by $q \mapsto \xi$. We can also restrict to $u^{\text{rest}} := u_\xi/(e^p, F^p, K^{2p} - 1)$; then $K^n$ is central.

Radford showed that there exists a unique $\mu^\xi \in H^\times$, called the integral, such that
\begin{equation}
(0.2) \quad (\mu \otimes \text{id})\Delta(x) = \mu(x)1
\end{equation}
for all $x \in H$. For example, in $u_\xi$,
\begin{equation}
(0.3) \quad \mu(E^m F^n K^\ell) = \delta_{m,p-1} \delta_{n,p-1} \delta_{\ell,p+1}.
\end{equation}

**Theorem 0.4** (Hennings, Kauffman-Radford). Let $L$ be a framed link and $M := S^3(L)$. Let $\sigma_+$, resp. $\sigma_-$, be the number of positive, resp. negative eigenvalues of $\ell_k(L)$. Then the Hennings invariant can be calculated as
\begin{equation}
(0.5) \quad \text{Hen}(M) = \frac{\mu^{\otimes |L|}(J_L)}{\mu(J_{U^+_+})^{\sigma_+} \mu(J_{U^-_-})^{\sigma_-}}.
\end{equation}

The proof is surprisingly simple.

**Proof.** Using the Kirby move $K_1$, write $L' = L \amalg u_+$, where $u_+$ indicates an unlinked unknot. Then
\begin{equation}
(0.6) \quad \text{Hen}(M) = \frac{\mu^{\otimes |L|}(J_L)\mu(J_{U^+_+})}{\mu(J_{U^+_-})^{\sigma_+} \mu(J_{U^-_-})^{\sigma_-}},
\end{equation}
and we cancel out the factors of $\mu(J_{U^+_+})$ in the numerator and denominator. Now, perform the Kirby move $K_2$ and use (0.2) and we’re done. 

**Remark 0.7.** If $H$ is semisimple,
\begin{equation}
(0.8) \quad \mu = \sum q\text{dim}(V_i) \text{tr}_q V_i,
\end{equation}
where the sum is over the isomorphism classes of irreducible modules $V_i$, and the Hennings invariant for $H$ and $M$ and the Witten-Reshetikhin-Turaev invariant for $H$ and $M$ coincide.

Kuperberg [Kup91] constructed a related invariant using cointegrals in a Hopf algebra.

**Definition 0.9.** Let $H$ be a Hopf algebra. A left cointegral is an element $c \in H$ satisfying $xc = \epsilon(x)c$ for all $x \in H$; a right cointegral satisfies $cx = \epsilon(x)c$ for all $x \in H$.

A Hopf algebra in which left and right cointegrals coincide is called unimodular.

Kuperberg’s invariant is the simplest algebraic 3-manifold invariant one can define with Hopf algebras, and this lends it its usefulness — it will probably be one of the first things we fully categorify. For example, if we chose $u_\xi(\mathfrak{sl}_2)$, we’d need the data of the Borel part, $\langle K, E \rangle$. Chang-Cui [CC19] showed that the Kuperberg invariant for $M$ and $H$ coincides with the Hennings invariant for $M$ and $\mathcal{D}(H)$, the double of $H$, analogous to the relationship between Reshetikhin-Turaev invariants for a modular tensor category and Turaev-Viro invariants for its Drinfeld double.

**Theorem 0.10** (Chen-Kuppum-Srinivasan [CKS09]). If $b_1(M) = 0$, the Hennings invariant of $M$ is $|H_1(M)|\text{WRT}(M)$; otherwise, it vanishes.
Proof. Our proof sketch follows Habiro-Lê. Note: I (the notetaker) didn’t follow what was written on the board; sorry about that. I think what happened was: the Hennings invariant of $M$ ends up being

$$\sum_{i \in I} x_i \otimes y_i,$$

where $\{x_i\}$ and $\{y_i\}$ are both bases of $H$. This induces a Hopf pairing, sometimes called the quantum Killing form, by declaring $\langle x_i, y_j \rangle := \delta_{ij}$. Hence if $x = \sum a_i y_i$, $\langle x, x_i \rangle = a_i$.

Let $M$ be an integral homology sphere, so we can realize $M$ as surgery on a knot $K$ framed with framing $\pm 1$. Then $I_M = \langle r^{-1}, J_K \rangle$, where $r$ is a ribbon element in $H$. We claim that for all $x \in H$,

\[
\langle r^{-1}, x \rangle = \frac{\mu(x)}{\mu(r)},
\]

and that $\Delta(r) = (r \otimes r)M$. This is because $(\mu \otimes \text{id})\Delta(r) = \mu(r)1$, so

\[
\sum_i \mu(rx_i)y_i = \mu(r)1,
\]

and therefore

\[
r^{-1} = \sum_i \frac{\mu(rx_i)}{\mu(r)}y_i
\]

\[
\langle r^{-1}, x_i \rangle = \frac{\mu(rx_i)}{\mu(r)}.
\]

If $b_1(M) > 0$, $S^2 \times S^1 = S^3(U_0)$, where $U_0$ denotes an unknot, and $J_{U_0} = 1$, so the Hennings invariant is $\mu(1) = 0$. $\Box$

This seems to spell doom for non-semisimple invariants, but not all of them are killed. This leads one to introduce modified traces, following Geer-Patureau (2008), functions $t_P : \text{End} P \to k$, where $P$ is an $H$-module, such that $t_P(fg) = t_P(gf)$ and $t_P$ satisfies the partial trace property in Figure 1.

![Figure 1. The partial trace property.](image)

Remark 0.15. Now taking the invariant $J_K$ as usual, this kind of invariant lands in

\[
qHH_0(H) := H/\langle xy0S^2(y)x \mid x, y \in H \rangle.
\]

Theorem 0.17 (Beliakova-Blanchet-Gainutdinov [BBG17]). Let $H$ be a unimodular pivotal Hopf algebra. Then for $f \in \text{End}_H(H)$,

\[
\text{tr}_H(f) = \mu_g(f(1)),
\]

where $g$ is the pivotal element: $\mu_g(x) = \mu(gx)$. In particular, the modified trace is uniquely determined.

Then, Beliakova-Blanchet-Geer [BBG18] used this to define more invariants for a knot $K_P$ in a 3-manifold $M = S^3(L)$: the invariant is $(\mu_{\otimes L} \otimes t_P) J_{L \cup K_P}$. These invariants were extended to a TQFT by De Renzi, Geer, and Patureau-Mirand [DRGPM18].

In the last few minutes, we’ll discuss CGP invariants. Consider the Hopf algebra

\[
u_{\text{unrolled}} := \langle K, E, F, H \rangle/\langle E^p, F^p \rangle.
\]
Given $\lambda \in \mathbb{C}$, we have a $p$-dimensional irreducible $u_{unrolled}$-module $V_\lambda$, though thanks to some redundancy, really the classification is in $\mathbb{C}/2\mathbb{Z}$. This leads to an invariant of a manifold together with a cohomology class $\lambda \in H^n(M; \mathbb{C}/2\mathbb{Z})$, given by

$$CGP(M, \lambda) := \sum_{k=0}^{p-1} d^{mod}(V_{\lambda+2k})J_K(V_{\lambda+2k})$$

This is a surprisingly simple description of this kind of invariant, which is nice. At $p = 2$, this specializes to the Alexander polynomial and Reidemeister torsion. Blanchet, Costantino, Geer, and Patureau-Mirand [BCGPM16] extended this invariant to a TQFT in which the order of the Dehn twist is trivial. The $S$- and $T$-matrices of this TQFT are related to work of Gukov and collaborators.

References


MSRI Lecture 2 - 1

Quantum invariants of links and 3-manifolds

\{ framed \} \xrightarrow{\text{surgeries}} \{ \text{cup product} \} \xrightarrow{\text{3-manifolds}} \%

U^\text{rest} \xrightarrow{\text{f.d.}} U^\text{unrolled}

\{ \text{logarithmic} \} \xrightarrow{\text{integration}} \{ \text{Hennings '98} \}
\quad \text{B-Blandt-Geer 'R'}
\quad \text{Costantino-Geer '11}
\quad \text{Patureau (CGP)}

(1) Algebraic data for Hennings and BBB

\[ H \text{ f.d. ribbon Hopf algebra} \]

Main example: \( \mathbb{F} = \mathbb{C} \), \( U^\text{rest} = \langle e, f, K \rangle \)

\[ e = \frac{1}{2} (1 - E) = (v - v^{-1})E \]

\[ K^2 = 1 \Rightarrow 2p \text{ is 2p th root of unity} \]

\[ K^p \in \mathbb{Z}(U^\text{rest}) \text{ is invertible, } K^p = \pm 1 \text{ } \text{Small is when } K^p \text{ acts by } \pm 1 \]
Book: Radford on Hopf algebras

\[ \mu \in H^+ \quad (\mu \circ \text{id}) \Delta(x) = \mu(x) \quad \forall x \in H \]

Example: \( \mu(E^m F^n K^p) = \delta_{n,m} \delta_{p,1} \delta_{q,0} \delta_{p,1} \)

Thm: (Hennings, Kauffman-Radford) let \( H = S^3(L) \)

\[ K \quad \# \text{ link comm} \]

\[ \text{Heun}(H) = \frac{\mu(J_+^L)}{\mu(J_{u+}^+ \mu(J_{u-}^+) \mu(J_{u-})} \]

\[ 6^+ \# \text{ pos. eigenvalues of } \mu(H). \]

Proof:

K1: \( L' = L U U_+ \)

\[ \frac{\mu \otimes 1_H(J_+)}{\mu(J_+^L \mu(U_+) \mu(J_{u+}^+) \mu(J_{u-}^+) \mu(J_{u-})} \]

K2:

\[ \mu(x) 1 = (\mu \circ \text{id}) \Delta(x) \]
Remark: 

Kuperberg using integral and coinTEGRAL C = H

1. \( x \cdot c = e(x)c \) left
2. \( c \cdot x = e(x)c \) right

H is unimodular if left image is right.

Involutive $S^2$ id.

Advantage: minimal algebraic data, no braiding and ribbon structure required.

Borel subalgebra works.

\[ \text{Kup} (H, H) = \text{Hen} (M, D(H)) \]

\[ \text{Kup} (M) = \text{Hen} (M \# - H) \]

If $H$ is semisimple (Kerler) $1995$

\[ \text{WRT} (M) = \text{Hen} (M) \]

\[ \text{Thm} (\text{Chen - Kuppad - Srinivasan}) 1997 \]

\[ \text{Hen} (M) = \left\{ \begin{array}{ll}
\text{WRT}(M) & \text{if } \delta_1(M) = 0 \\
0 & \text{if } \delta_1(M) > 0
\end{array} \right. \]
Proof: (Habiro-Le): $\mathcal{U}_{\text{root}}$ is factorizable.

$\Rightarrow \quad N = \mathbb{F}_2 \mathcal{R} = \sum x_i \otimes y_i \quad \{ x_i, y_i \} \text{ are base of } \mathcal{U}_{\text{root}}$

Hodge pairing
(quantum Killing form)

$< x_i, y_j > = \delta_{ij}$

$\Rightarrow \quad x = \sum a_i y_i = \sum < x_i x_j y_j >$

Let $M$ be a 2HS
The unified Habiro's invariant of $M$

$I_M = < r', j >$

Claim: $< r', x > = \frac{\mu(xr)}{\mu(r)} \quad \forall x \in H$

Indeed,

$r' = (\mu \otimes \text{id}) \Delta r = \mu(r) \Delta = \sum_{i} \mu(r x_i) y_i$

$\Delta r = (r \otimes r) H$

$r' = \sum \frac{\mu(r x_i)}{\mu(r)} y_i$

$< r', x_i > = \frac{\mu(r x_i)}{\mu(r)}$
Case $b_1 > 0$

$S^2 \times S^2 = S^3$ of $U$ \quad $\mathbb{V} = 1$ \quad $\mu(J_{\mathbb{V}}) = 0$

Q: Does it kill the non-semisimple story?

A: No:

Improvement: \quad $H = \oplus P_i$

$$\begin{array}{c}
P \quad \mathbb{F} \quad P \\
0 \quad \mathbb{F} \quad P
\end{array}$$

Solution: Modified trace

\begin{align*}
\{ t_p : \text{End } P &\to \mathbb{K} \} & \quad P \in H\text{-mod} \\
. t_p (fg) &= t_p (gf) \\
. t_{p_v} \left( \mathbb{F} \begin{array}{c} 1 \\ P \end{array} \right) &= t_p \left( \mathbb{F} \begin{array}{c} 1 \\ V \end{array} \right)
\end{align*}

Remark: $2$ Knots $\xrightarrow{J_k}$ $H \xrightarrow{\text{cat-trace}} \mathbb{K}$

$\xrightarrow{\text{cat-trace}}$ $\mathbb{K}$
\[ q_{H^2}(\langle H \rangle) = H \]
\[ \{ xy - y s^2(x) \mid x, y \in H \} \]

factorizes through \( H^2_0 \langle H \rangle \), but is compatible with categorical trace

2. \( \text{BBG invariant} \quad \mathcal{M} = \mathcal{S}^2(L) \)
\[ \text{BBG}(M, K_p) = \left( \mu \otimes \kappa^1 \otimes \tau_p \right) \left( J_{\mathbb{L}, \mathbb{K}_p} \right) \]

non-zero if \( b_1 > 0 \), if we add an ununit.

1. \( \text{Thm (\textit{B}-Blanchet - Gainetdinov)} \)

A unimodular pivotal \( H \)
\[ t_H(f) = \mu_{\mathcal{S}}(f(1)) \quad \forall f \in \text{End}_H \, H \]

9 pivotal element
\[ \mu_{\mathcal{S}}(x) = \mu_{\mathcal{S}}(g x) \] 
this form has a partial trace property suitable to define non-semisimple invariants

Corol.

\[ (\Rightarrow) \] \( H \)-mod is semisimple (unimodular pivotal)
\[ (\Leftarrow) \] its categorical trace \( \neq 0 \) then \( t^c = \mu_{\mathcal{S}} \).
3. \( CGP \) unrolled: \( \mathcal{U} \rightarrow \mathcal{U}^{1/p} / \mathbb{C}^{1/p} > \)

What happens? \( Ku = \lambda u \) if \( k_{1/p} = 1 \Rightarrow \lambda = 1 \)

We have \( \lambda \in \mathbb{C} ! \)

Irrep proj. \( V_{\lambda} \) of dim \( p \) for \( \lambda \in \mathbb{C} \)

\( \lambda \in H^1(M, \mathbb{C}/2\pi) \)

\( t_{\lambda} \left( i\Phi_{\lambda} \right) = d(V_{\lambda}) \) modified dimension

\( \text{CGP} (M, \lambda) := \sum_{k=0}^{p-1} d(V_{\lambda + 2k}) \) \( \text{J}_k \left( V_{\lambda + 2k} \right) \) (up to normalization)

Properties:
1. Extends to \( TAFT \)
2. The order of the twist is \( \infty \)
3. Gunay & observe \( ST \) matrices from \( \text{TAFT} \) in their construction.
4. On group project with C. Blanchet

We are relating \( \text{J}_k \left( V_{\lambda + 2k} \right) \) to a deformed cyclotomic expansion of Habiro and hope for a universal version of these results with Chenevix and Blanchet.