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Introduction to higher categories, dualizability, and applications to topological field theories

This is a long selection of exercises of very different levels and with motivations coming from different areas. I am aware that this list is too long for the problem sessions. Pick the one(s) you find interesting and look up or ask for the precise definitions if needed.

(1) Find the dualizable objects in the following monoidal categories:

(a) vector spaces and direct sum

(b) vector spaces and tensor product

(c) pointed vector spaces (a vector space together with a chosen vector in it), point-preserving linear maps, and tensor product

(d) sets and cartesian product

(e) Span, where objects are sets, a morphism from X to Y is an isomorphism class of spans $X \leftarrow S \rightarrow Y$, composition is pullback, and the monoidal product is the cartesian product

(f) Alg, where objects are $\mathbb{C}$-algebras, a morphism from an algebra A to an algebra B is an isomorphism class of bimodules, composition is relative tensor product,

$$_BN_C \circ_A M_B =_A M_B \otimes_B {}_BN_C;$$

and tensor product over $\mathbb{C}$ as the monoidal structure

(g) nCob and disjoint union

(2) Show that if Z is an n-dimensional topological field theory, then for any closed $(n-1)$-dimensional manifold, $Z(M)$ is finite dimensional.

(3) (a) Show that any small category with a single object is the same data as a monoid.

(b) Let A be a set. Which structure does A need to have so that there is a 2-category with a single object, a single morphism, and A as the set of 2-morphisms?

(4) Look up the details of the definition of a quasi-category. Show the following properties:

(a) Translate the horn-filling conditions for Kan complexes and quasi-categories in dimensions 1, 2, and 3 into categorical content.

(b) Let $\tau_1 \colon sSet \to Cat$ be the left adjoint to the nerve functor, called *homotopy category*. Work out/look up an explicit description of τ_1.

(5) (a) Which 1-morphisms have left and/or right adjoints in the following bicategories or $(\infty, 2)$-categories:

(i) Alg^{bi} (*Hint: Look up and use the dual basis lemma from commutative algebra.)*

(ii) Span_2^{bi}

(b) Which objects are 2-dualizable in the following symmetric monoidal bicategories or $(\infty, 2)$-categories (we haven't seen these in detail, but try to figure out the pictures):

(i) $2\mathrm{Cob}^{ext}$ and $2\mathrm{Cob}^{ext,fr}$

(ii) Bord_2 and $\mathrm{Bord}_2^{\mathrm{fr}}$

(6) Let $\mathcal{C}$ be a monoidal category.

   (a) Show that we can define a bicategory $\Sigma\mathcal{C}$ with a single object, with the objects of $\mathcal{C}$ as 1-morphisms, and the morphisms in $\mathcal{C}$ as 2-morphisms.

   (b) Think about the converse situation: given a bicategory $\mathcal{B}$ with one object $\star$, which structure does the category $\Omega_\star\mathcal{B} = \mathrm{End}_\mathcal{B}(\star)$ have?

   (c) Show that an object in $\mathcal{C}$ is left/right dualizable if and only if it has a left/right adjoint in $\Sigma\mathcal{C}$.

(7) (a) Show that a strict monoidal category $\mathcal{C}$ determines a functor $\mathcal{C}^\otimes : \Delta^{op} \longrightarrow \mathrm{Cat}$ such that, for every $n \geq 0$, the maps induced by the inclusions $[1] \to [n]$, $0 \mapsto i-1, 1 \mapsto i$, for $1 \leq i \leq n$, are equivalences:

$$\mathcal{C}_n^\otimes \xrightarrow{\simeq} \left(\mathcal{C}_1^\otimes\right)^{\times n}. \tag{1}$$

   Show that this assignment extends to a functor. What happens if we start with a not necessarily strict monoidal category?

   (b) Now start with a strict symmetric monoidal category $\mathcal{C}$. Construct a functor

$$\mathcal{C}^\otimes : \mathrm{Fin}_* \longrightarrow \mathrm{Cat},$$

   with a similar condition as in (1). Here Fin_* is the category of finite pointed sets and pointed maps. Viewing $\mathcal{C}$ as just a monoidal category we also get a functor from 7a. How do these compare? (This serves as a justification for using the same symbol for either functor.)

(8) (a) Show that in an $(\infty, 2)$-category, a 2-morphism which has a right (or left) adjoint necessarily is invertible. What is the analogous statement for (∞, n)?

   (b) Conclude that an n-dualizable object in an (∞, k)-category, for $k < n$, is invertible. What does this imply for fully extended TFTs?

   (c) Show that the image of an n-dualizable object under a symmetric monoidal functor is n-dualizable. What does this imply for fully extended TFTs?

(9) (a) Using exercises 1f and 5(a)i, find the 2-dualizable objects in Alg^{bi}. (We have not worked out the details of the symmetric monoidal structure on Alg^{bi}. It is given by tensor product over $\mathbb{C}$. Showing that this indeed gives a *symmetric* monoidal structure is tedious.)

   (b) Show that for any finite group G, the group algebra $\mathbb{C}[G]$ is 2-dualizable.

   (c) From 9b we know that there is a framed fully extended 2 dimensional TFT which sends the point to $\mathbb{C}[G]$. What is its value on S^1? *(Hint: It might help to think about the general case.)*

   (d) *Extra:* In fact, it gives an oriented 2TFT. What is its value on a closed surface? You might like to relate this to Nathalie Wahl's minicourse.