1) Prove that the composite of two adjunctions in a 2-category $K$ is again an adjunction. In other words, given

```
A ⊲ - □ - ⊳ B ⊲ - □ - ⊳ C ⊲ - □ - ⊳ K
```

\[ \eta : \id_B \Rightarrow u \circ f \]
\[ \varepsilon : f \circ u \Rightarrow \id_A \]
\[ \mu : \id_C \Rightarrow v \circ g \]
\[ \nu : g \circ v \Rightarrow \id_B \]

Write down candidates for unit and counit for an adjunction $f g \dashv u \circ v$ and show they satisfy the triangle identities.

**Hint** Try using string diagrams to express / plan your argument.

2) Given an adjunction $E : A \dashv \Sigma : B \dashv \Pi$ in a 2-category $K$, show that it gives rise to adjunctions of hom-categories.

\[ K(X, f) \dashv K(X, u) + K(u, y) \dashv K(f, y) \]

\[ f a \Rightarrow b \]
\[ a \Rightarrow u b \]

\[ c u \Rightarrow \delta f \]
\[ c \Rightarrow d f \]

Hmmm a little word!
3) **Given an equivalence in a 2-category K**

\[ A \xleftarrow{\omega} \xrightarrow{\eta} B \]

\[ \eta : \omega \circ \eta = Id_B \]

\[ \beta : Id_A = \omega \circ \eta \]

Show that we may construct a 2-cell \( \beta' : Id_A \Rightarrow \omega \circ \eta \) such that \( \eta \) and \( \beta' \) satisfy the triangle identities.

**Hint:** Use sequences of manipulations of string diagrams to plan your argument (Nomadic Math) then convert those into algebraic equivalences in which each step is justified by epimod by middle 4 interchange or by an isomorphism identity (Royal Math).
4) Adapt the proof given in (3) to prove the following slightly more general result.

Suppose we are given 1-cells \( A \xleftarrow{\varepsilon} B \) in a 2-category \( K \) along with 2-cells \( \gamma : \varepsilon_B \Rightarrow uf + \varepsilon : fu \cong \varepsilon_A \) such that \( \varepsilon_B \) and \( \gamma u \) are both isomorphisms. Then we may construct a 2-cell \( \gamma' : \varepsilon_B \Rightarrow uf \) such that this pair \( \varepsilon, \gamma' \) satisfy the triangle identities and thus display an adjunction \( f \dashv u \).

**Left Adjoint Right Inverse (LARI)**

This class of adjunctions is surprisingly handy.
5) Prove that in order to show that a pair of 1-cells \( A \xrightarrow{f} B \) are adjoint it suffices to give 2-cells \( \eta : \text{id}_B \Rightarrow uf \) and \( \epsilon : fu \Rightarrow \text{id}_A \) such that the triangle identities
\[
\begin{align*}
  f & \Rightarrow fu \Rightarrow uf \Rightarrow f & \text{and} \\
  u & \Rightarrow ufu \Rightarrow uf & \Rightarrow u
\end{align*}
\]
are both isomorphisms.

6) Prove the following result stated in Part II:
Right adjoints preserve limits of families

\[
\begin{array}{ccc}
  A & \xleftarrow{\perp} & B \\
  \downarrow f & & \downarrow u \\
  A^x & \xrightarrow{\perp} & B^x
\end{array}
\]

Given an adjunction

If
\[
\begin{array}{ccc}
  I & \xrightarrow{l} & A \\
  \downarrow \lambda & & \downarrow \Delta \\
  I & \xrightarrow{d} & A^x
\end{array}
\]
is an absolutes right lifting

\[\Rightarrow\]

\[
\begin{array}{ccc}
  I & \xrightarrow{l} & A \\
  \downarrow \lambda & & \downarrow \Delta \\
  I & \xrightarrow{d} & A^x
\end{array}
\]
is also an absolutes right lifting.
7) Use the hom-wise characterisation of adjunctions and the well known equivalence between the thin and fat slices of a quasi-category to complete the exercise on slide 25, viz:

Suppose that $A$ is a quasi-category; an object in the oo-cosmos $\mathcal{QC}_\infty$. Let $t$ be a vertex of $A$, then

$$\varepsilon : 1 \rightarrow A$$ is a terminal element

This is Joyal's terminal object notion

$A$ admits fillers for simplex boundaries with last vertex $t$
7) Use the result cited on slide 36, 1/3:

**Key Observation** It is a consequence of this weak 2-universal property of $\text{Hom}_c(f,g)$ that there exists a bijection:

$$
\begin{align*}
&\xymatrix{ 
X & \ar[l]_a & A \\
B & \ar[u]^b & C \\
& \ar[ll]^c}
\end{align*}
$$

2-cells

$$
\begin{align*}
&\xymatrix{ 
X \ar[r]^f & \text{Hom}_c(f,g) \\
& \ar[l]_{(b,c)} \ar[u]^{(b_0, g_0)} \ar[d]_{g} & A \\
& \ar[u]^{g_0} & B}
\end{align*}
$$

Iso-classes of 1-cells over $B \times A$

To prove this hom-wise characterization of absolute right liftings:

**Lemma** The triangle on the left is an absolute right lifting if and only if the induced functor $\text{Hom}_A(A_1, L) \to \text{Hom}_C(f, g)$ is an equivalence.