Strong Szegő Theorem on a Jordan Curve

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Dedicated to the memory of Harold Widom 1932–2021

Introduction

γ a Jordan curve in $\mathbb{C}$ and $g : \gamma \mapsto \mathbb{C}$ a given function.

Consider the **determinant**

$$D_n[e^g] = \det \left(\int_\gamma \zeta^j \bar{\zeta}^k e^{g(\zeta)} \, |d\zeta| \right)_{0 \leq j,k < n}$$

$\gamma = \mathbb{T}$, the unit circle gives Toeplitz determinants. Related to orthogonal polynomials on γ with weight e^g if the weight is positive.

Introduced by Szegő in a paper from 1921 in the case $g = 0$:

Über orthogonale Polynome, die zu einer gegebenen Kurve der komplexen Ebene gehören, Math. Z. **9** (1921), 218–270

See also the last chapter in Szegő's book *Orthogonal polynomials*.

Introduction

For Toeplitz determinants we have the **strong Szegő limit theorem** which gives precise asymptotics for D_n.

Want to generalize to other Jordan curves.

Another interpretation: **Planar Coulomb gas** on the curve

$$D_n[e^g] = \frac{1}{n!}\int_{\gamma^n} \prod_{1\le\mu\ne\nu\le n} |\zeta_\mu - \zeta_\nu| \prod_{\mu=1}^{n} e^{g(\zeta_\mu)} \prod_{\mu=1}^{n} |d\zeta_\nu|$$
$$= \frac{1}{n!}\int_{\gamma^n} e^{-\sum_{\mu\ne\nu}\log|\zeta_\mu-\zeta_\nu|^{-1}+\sum_\mu g(\zeta_\mu)}|d\zeta|.$$

In particular

$$Z_n(\gamma) := D_n(1)$$

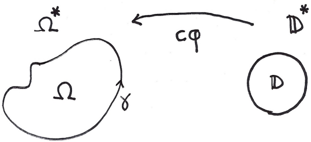
is the **partition function**. Asymptotics?

Szegő theorem on a Jordan curve

Let Ω^* be the unbounded part of the complement of γ and $\mathbb{D}^*$ the exterior of the closed unit disk.

Let $c\phi : \mathbb{D}^* \mapsto \Omega^*$ ($c =$ the capacity of γ) be the **exterior Riemann mapping function**.

$$\phi(z) = z + \phi_0 + \phi_{-1} z^{-1} + \dots$$



Leading order asymptotics as $n \to \infty$

$$D_n[e^g] \sim \exp\left(-n^2 \inf_{\mu} \int_{\gamma} \int_{\gamma} \log |\zeta_1 - \zeta_2|^{-1} \, d\mu(\zeta_1) d\mu(\zeta_2) \right)$$
$$= \exp(-n^2 V(\gamma)) = \operatorname{cap}(\gamma)^{n^2}.$$

Szegő theorem on a Jordan curve

Let $|z| > 1, |\zeta| > 1$. We have the expansion

$$\log \frac{\phi(\zeta) - \phi(z)}{\zeta - z} = - \sum_{k,\ell=1}^{\infty} a_{k\ell} \zeta^{-k} z^{-\ell},$$

$a_{k\ell}$ **Grunsky coefficients**, $a_{k\ell} = a_{\ell k}$.

If γ is a quasicircle there is a $\kappa < 1$ so that we have the **Grunsky inequality**

$$\left| \sum_{k,\ell=1}^{\infty} \sqrt{k\ell}\, a_{k\ell} w_k w_\ell \right| \leq \kappa \sum_{k=1}^{\infty} |w_k|^2,$$

Szegő theorem on a Jordan curve

$$\log \frac{\phi(\zeta) - \phi(z)}{\zeta - z} = - \sum_{k,\ell=1}^{\infty} a_{k\ell} \zeta^{-k} z^{-\ell},$$

Let

$$B = (b_{k\ell}) = (\sqrt{k\ell} a_{k\ell}) = (b_{k\ell}^{(1)}) + \mathrm{i}(b_{k\ell}^{(2)}) = B_k^{(1)} + \mathrm{i} B_k^{(2)}$$

be the **Grunsky operator** on $\ell^2(\mathbb{C})$. It is a complex and symmetric infinite matrix.

Szegő theorem on a Jordan curve

Let

$$B = (b_{k\ell}) = (\sqrt{k\ell}a_{k\ell}) = (b^{(1)}_{k\ell}) + \mathrm{i}(b^{(2)}_{k\ell}) = B^{(1)}_k + \mathrm{i}B^{(2)}_k$$

be the **Grunsky operator** on $\ell^2(\mathbb{C})$. It is a complex and symmetric infinite matrix.

Define

$$K = \begin{pmatrix} B^{(1)} & B^{(2)} \\ B^{(2)} & -B^{(1)} \end{pmatrix}.$$

on $\ell^2(\mathbb{C}) \oplus \ell^2(\mathbb{C})$ which is real and symmetric.

We have the Fourier expansion

$$g(\phi(e^{\mathrm{i}\theta})) = \frac{a_0}{2} + \sum_{k=1}^{\infty} a_k \cos k\theta + b_k \sin k\theta.$$

Szegő theorem on a Jordan curve

Write

$$\mathbf{g} = \begin{pmatrix} (\frac{1}{2}\sqrt{k}a_k)_{k\geq 1} \\ (\frac{1}{2}\sqrt{k}b_k)_{k\geq 1} \end{pmatrix} \in \ell^2(\mathbb{C}) \oplus \ell^2(\mathbb{C}).$$

Theorem

Assume that γ is $C^{5+\alpha}$, $\alpha > 0$, and that

$$\sum_{k=1}^{\infty} k(|a_k|^2 + |b_k|^2) < \infty.$$

Then

$$D_n[e^g] = \frac{(2\pi)^n \mathrm{cap}(\gamma)^{n^2}}{\sqrt{\det(I+K)}} \exp\left(na_0/2 + \mathbf{g}^t(I+K)^{-1}\mathbf{g} + o(1)\right),$$

as $n \to \infty$.

Optimal conditions on g. Not optimal for γ. More about the case $g = 0$ later.

Example

Let γ be an **ellipse** with half-axes $1+\rho^2$ and $1-\rho^2$; then $\mathrm{cap}(\gamma)=1$,

$$\phi(z) = z + \rho^2/z \quad \text{and} \quad b_{k\ell} = \rho^{k+\ell}\delta_{k\ell}.$$

In this case

$$D_n[e^g] = \frac{(2\pi)^n}{\prod_{k=1}^{\infty}(1-\rho^{4k})^{1/2}} \exp\left(\frac{na_0}{2} + \frac{1}{4}\sum_{k=1}^{\infty} k\left(\frac{a_k^2}{1+\rho^{2k}} + \frac{b_k^2}{1-\rho^{2k}}\right) + o(1)\right)$$

as $n \to \infty$.

Earlier results

Szegő and Grenander-Szegő proved for analytic γ

$$\lim_{n\to\infty} \mathrm{cap}(\gamma)^{-2n-1} \frac{D_{n+1}[e^g]}{D_n[e^g]} = 2\pi \exp\left(\frac{1}{2\pi} \int_{-\pi}^{\pi} g(\phi(e^{\mathrm{i}\theta}))\, d\theta \right).$$

I proved the following **relative Szegő theorem** in my thesis (J. '88)

$$\frac{D_n[e^g]}{D_n[1]} = \exp[na_0/2 + \mathbf{g}^t(I+K)^{-1}\mathbf{g} + o(1)]$$

as $n \to \infty$ under stronger assumptions than in the new theorem.

Heuristic proof

Assume w.l.o.g. that $\mathrm{cap}(\gamma) = 1$ and $a_0 = 0$.

$$\frac{1}{(2\pi)^n} D_n[e^g]$$

$$= \frac{1}{(2\pi)^n n!} \int_{[-\pi,\pi]^n} \prod_{\mu \neq \nu} \left| \frac{\phi(e^{i\theta_\mu}) - \phi(e^{i\theta_\nu})}{e^{i\theta_\mu} - e^{i\theta_\mu}} \right| \prod_\mu e^{\sum_\mu g(\phi(e^{i\theta_\mu})) + \log|\phi'(e^{i\theta_\mu})|}$$

$$\times \prod_{\mu \neq \nu} \left| e^{i\theta_\mu} - e^{i\theta_\mu} \right| \, d\theta$$

$$= \mathbb{E}_n \left[\exp\left(-\operatorname{Re} \sum_{k,\ell=1}^{\infty} a_{k\ell} \left(\sum_\mu e^{-ik\theta_\mu} \right) \left(\sum_\nu e^{-i\ell\theta_\nu} \right) + \sum_\mu g(\phi(e^{i\theta_\mu})) \right) \right]$$

$$= \lim_{m \to \infty} \mathbb{E}_n \left[\exp\left(-\operatorname{Re} \sum_{k,\ell=1}^{m} b_{k\ell} \left(\frac{1}{\sqrt{k}} \sum_\mu e^{-ik\theta_\mu} \right) \left(\frac{1}{\sqrt{\ell}} \sum_\nu e^{-i\ell\theta_\nu} \right) \right.\right.$$

$$\left.\left. + \sum_\mu g(\phi(e^{i\theta_\mu})) \right) \right].$$

Heuristic proof

Introduce the infinite column vectors

$$\mathbf{X} = \left(\frac{1}{\sqrt{k}} \sum_{\mu} \cos k\theta_\mu \right)_{k\geq 1}, \quad \mathbf{Y} = \left(\frac{1}{\sqrt{k}} \sum_{\mu} \sin k\theta_\mu \right)_{k\geq 1},$$

We have the expression

$$-\operatorname{Re} \sum_{k,\ell=1}^{m} b_{k\ell}(x_k - \mathrm{i}y_k)(x_\ell - \mathrm{i}y_\ell) = -\begin{pmatrix} P_m\mathbf{X} \\ P_m\mathbf{Y} \end{pmatrix}^t K_m \begin{pmatrix} P_m\mathbf{X} \\ P_m\mathbf{Y} \end{pmatrix}$$

$$= -\begin{pmatrix} P_m\mathbf{X} \\ P_m\mathbf{Y} \end{pmatrix}^t T_m \begin{pmatrix} -\Lambda_m & 0 \\ 0 & \Lambda_m \end{pmatrix} T_m^t \begin{pmatrix} P_m\mathbf{X} \\ P_m\mathbf{Y} \end{pmatrix},$$

where

$$K_m = \begin{pmatrix} B_m^{(1)} & B_m^{(2)} \\ B_m^{(2)} & -B_m^{(1)} \end{pmatrix}, \quad \Lambda_m = \operatorname{diag}(\lambda_{m,1}, \ldots, \lambda_{m,m}),$$

T_m is an orthogonal matrix, and $\lambda_{m,k}$ are the singular values of B_m. By Grunsky's inequality $|\lambda_{m,k}| \leq \kappa < 1$.

Heuristic proof

Let **u** and **v** be two real column vectors in $\mathbb{R}^m$. Set

$$L_m = \begin{pmatrix} P_m & 0 \\ 0 & P_m \end{pmatrix}^t T_m \begin{pmatrix} \mathrm{i}\Lambda_m^{1/2} & 0 \\ 0 & \Lambda_m^{1/2} \end{pmatrix} \begin{pmatrix} \mathbf{u} \\ \mathbf{v} \end{pmatrix}.$$

Then our formulas give

$$\frac{1}{(2\pi)^n} D_n[e^g] = \lim_{m\to\infty} \frac{1}{\pi^m} \int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv e^{-u^t u - v^t v} \mathbb{E}_n \left[\exp(2(L_m + \mathbf{g})^t \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} \right].$$

We want to take the limit $n \to \infty$.

Heuristic proof

Then our formulas give

$$\frac{1}{(2\pi)^n} D_n[e^g] = \lim_{m\to\infty} \frac{1}{\pi^m} \int_{\mathbb{R}^m} du \int_{\mathbb{R}^m} dv e^{-u^t u - v^t v} \mathbb{E}_n \left[\exp(2(L_m + \mathbf{g})^t \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} \right].$$

We want to take the limit $n \to \infty$. If we formally interchange the two limits and take the $n \to \infty$ limit inside the Gaussian integral we need to compute

$$\lim_{n\to\infty} \mathbb{E}_n \left[\exp(2(L_m + \mathbf{g})^t \begin{pmatrix} \mathbf{X} \\ \mathbf{Y} \end{pmatrix} \right]$$

which can be done using the strong Szegő limit theorem for Toeplitz determinants. Computing the Gaussian integral and letting $m \to \infty$ then gives

$$\lim_{n\to\infty} \frac{1}{(2\pi)^n} D_n[e^g] = \frac{1}{\sqrt{\det(I+K)}} \exp\left(\mathbf{g}^t (I+K)^{-1} \mathbf{g} + o(1) \right).$$

Steps in the real proof

- **Upper bound**. If f is real valued on $\mathbb{T}$ and $\hat{f}_0 = 0$, then

$$\mathbb{E}_n[e^{\sum_\mu f(e^{i\theta_\mu})}] \le e^{\sum_{k=1}^{\infty} k|\hat{f}_k|^2}.$$

- To get real-valued objects use analytic continuation and normal families.
- **Lower bound**. Change of variables $\theta_\mu = \phi_\mu - \frac{1}{n}h(\phi_\mu)$ plus Jensen's inequality and appropriate h.
- Grunsky part should not be too big. Leads to regularity assumptions on γ.

Asymptotics of the partition function

The theorem gives for $Z_n(\gamma) = D_n[1]$,

$$\lim_{n\to\infty} \log \frac{Z_n(\gamma)/\mathrm{cap}(\gamma)^{n^2}}{Z_n(\mathbb{T})/\mathrm{cap}(\mathbb{T})^{n^2}} = \lim_{n\to\infty} \log \frac{Z_n(\gamma)}{(2\pi)^n \mathrm{cap}(\gamma)^{n^2}} = -\frac{1}{2}\log\det(I - B^*B),$$

since $\det(I+K) = \det(I - B^*B)$.

The quantity $-\frac{1}{2}\log\det(I - B^*B)$ is, up to a multiplicative constant, the **Loewner energy** of the curve γ. It has also appeared as a Kähler potential for the Weil-Petersson metric on the universal Teichmüller space $T_0(1)$.

Curves with finite Loewner energy are called **Weil-Petersson quasicircles**. The curve γ is a Weil-Petersson quasicircle if and only if the Grunsky operator is Hilbert-Schmidt.

Asymptotics of the partition function

Some references on the Loewner energy and Weil-Petersson quasicircles:

Takhtajan, L. A., Teo, L.-P., *Weil-Petersson metric on the universal Teichmüller space*, Mem. Amer. Math. Soc. **183** (2006), no. 861

Wang, Y., *Equivalent descriptions of the Loewner energy*, Invent. Math. **218** (2019), no. 2, 573–621

Bishop, C. J., *Weil-Petersson curves, β-numbers and minimal surfaces*, http://www.math.stonybrook.edu/ bishop/papers/wpce.pdf

Viklund, F., Wang, Y., *Interplay between Loewner and Dirichlet energies via conformal welding and flow-lines*, Geom. Funct. Anal. **30** (2020) 289–321

A new characterization of Weil-Petersson quasicircles

Theorem

The Jordan curve γ is a Weil-Petersson quasicircle if and only if

$$\limsup_{n\to\infty} \frac{Z_n(\gamma)}{(2\pi)^n \mathrm{cap}(\gamma)^{n^2}} < \infty,$$

and in that case we have the limit

$$\lim_{n\to\infty} \log \frac{Z_n(\gamma)}{(2\pi)^n \mathrm{cap}(\gamma)^{n^2}} = -\frac{1}{2} \log \det(I - B^* B).$$

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Does not follow from above. Let γ_r be given by $\frac{1}{r}\phi(rz)$, $r > 1$, an analytic curve. Use the fact that $\frac{Z_n(\gamma_r)}{(2\pi)^n \mathrm{cap}(\gamma)^{n^2}}$ is increasing in n and decreasing in r which of course has to be proved.

Thank you for your attention!