Hankel composition structures in random matrix theory and beyond

Thomas Bothner

School of Mathematics
University of Bristol

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Inspired by works of

Figure 1: Pierre Le Doussal, Alexandre Krajenbrink, Satya Majumdar, Gregory Schehr.

and based on work in progress by the speaker.

What's the problem?

Consider the Gaussian Orthogonal Ensemble (GOE), i.e. matrices

$$\mathbf{X} = \frac{1}{2}(\mathbf{Y} + \mathbf{Y}^\top) \in \mathbb{R}^{n \times n}: \;\; Y_{ij} \overset{\text{iid}}{\sim} N(0,1).$$

(Hsu 1939; Wigner 1955; Mehta 1960)

It is known that, as $n \to \infty$,

$$\max_{i=1,\ldots,n} \lambda_i(\mathbf{X}) \Rightarrow \sqrt{2n} + \frac{F_1}{\sqrt{2}n^{1/6}},$$

(Bronk 1964; Mehta 1971; Forrester 1993)

Tracy, Widom 1996

$$\mathbb{P}(F_1 \leq t) = \exp\left[-\frac{1}{2}\int_t^\infty (s-t)(q(s))^2\,\mathrm{d}s - \frac{1}{2}\int_t^\infty q(s)\,\mathrm{d}s\right] \quad (1)$$

and $q = q(s)$ solves an ODE boundary value problem

Consider the Real Ginibre ensemble (GinOE), i.e. matrices

$$\mathbf{X} = \mathbf{Y} \in \mathbb{R}^{n \times n} : \quad Y_{ij} \overset{\text{iid}}{\sim} N(0,1). \quad \text{(Ginibre 1965)}$$

It is known that, as $n \to \infty$,

$$\max_{\substack{i=1,\ldots,n \\ \lambda_i \in \mathbb{R}}} \lambda_i(\mathbf{X}) \Rightarrow \sqrt{n} + \chi, \quad \text{(Rider, Sinclair 2014)}$$

Baik, Bothner 2018

$$\mathbb{P}(\chi \leq t) = \exp\left[-\frac{1}{2}\int_t^\infty (s-t)(p(s))^2\,\mathrm{d}s - \frac{1}{2}\int_t^\infty p(s)\,\mathrm{d}s\right] \qquad (2)$$

and $p = p(s)$ is a bit more complicated.

The limiting cdfs (1) (GOE) and (2) (GinOE) look suspiciously similar, at least on the structural surface. Perhaps a coincidence?

Probabilistic Universality

The above limit laws are universal in the class of Wigner random matrices (Soshnikov 1999) and in the class of real non-Hermitian random matrices with iid entries (Cipolloni, Erdös, Schröder 2019).

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Thin out the Pfaffian point processes $\{\lambda_i(\mathbf{X})\}_{i=1}^{n}$ and $\{\lambda_i(\mathbf{X}) \in \mathbb{R}\}_{i=1}^{m_n}$ by discarding each λ_i independently with likelihood $1-\gamma \in [0,1]$.

Q: Can we compute statistical properties of the resulting point processes $\{\lambda_i(\mathbf{X})\}_{i=1}^{n_\gamma}$ and $\{\lambda_i(\mathbf{X}) \in \mathbb{R}\}_{i=1}^{m_{n,\gamma}}$?

More evidence for structural universality

Dieng 2005; Bothner, Buckingham 2018 ($\bar{\gamma} := \gamma(2-\gamma) \in [0,1]$)

$$(t,\gamma) \in \mathbb{R} \times [0,1]: \quad \mathbb{P}(F_{1,\gamma} \leq t) = \exp\left[-\frac{1}{2}\int_t^\infty (s-t)(q(s;\bar{\gamma}))^2\,\mathrm{d}s\right] \times$$

$$\times \sqrt{\frac{\gamma - 1 - \cosh\mu(t;\bar{\gamma}) + \sqrt{\bar{\gamma}}\sinh\mu(t;\bar{\gamma})}{\gamma - 2}}, \quad \mu(t;\gamma) := \int_t^\infty q(s;\gamma)\,\mathrm{d}s.$$

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Baik, Bothner 2020

$$(t,\gamma) \in \mathbb{R} \times [0,1]: \quad \mathbb{P}(\chi_{\gamma} \leq t) = \exp\left[-\frac{1}{2}\int_t^{\infty}(s-t)(p(s;\bar{\gamma}))^2\,\mathrm{d}s\right] \times$$

$$\times\sqrt{\frac{\gamma - 1 - \cosh\nu(t;\bar{\gamma}) + \sqrt{\bar{\gamma}}\sinh\nu(t;\bar{\gamma})}{\gamma-2}}, \quad \nu(t;\gamma) := \int_t^{\infty} p(s;\gamma)\,\mathrm{d}s.$$

Why so similar?

In obtaining the above exact formulæ, one typically

(1) starts from the model's finite n correlation functions,

(2) computes a finite n gap probability as operator determinant,

(3) then passes to a suitable large n limit.

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For GOE and GinOE the *limiting* distribution functions equal

$$D(t,\gamma) \text{ “}=\text{” } \sqrt{\det(I-\bar{\gamma}K_t-\gamma\phi_t\otimes\varphi_t\restriction_{L^2(\mathbb{R}_+)})},$$

where K_t is a Hankel composition operator with kernel

$$K_t(x,y)=\int_0^\infty \phi_t(x+z)\psi_t(z+y)\,\mathrm{d}z;\quad f_t(x):=f(x+t). \tag{3}$$

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Precisely, $\phi(x)=\psi(x)=\mathrm{Ai}(x)$ and $\phi(x)=\psi(x)=\mathrm{e}^{-x^2}/\sqrt{\pi}$.

Once the structure (3) has been flushed out one attempts to massage it in integrable shape, i.e. one tries to find $f_j, g_j \in L^\infty(\mathbb{R}_+)$ such that

$$\frac{\sum_{j=1}^N f_j(x)g_j(y)}{x-y} = K_t(x,y) = \int_0^\infty \phi_t(x+z)\psi_t(z+y)\,\mathrm{d}z. \tag{4}$$

This can (e.g. GOE) or cannot (e.g. GinOE) work out, see (Blower 2008), but is considered in general desirable given that integrable operators share many remarkable properties:

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> stable under composition, resolvent of same type and accessible via Riemann-Hilbert problem $\rightarrow$ dynamical systems, asymptotics (Its, Izergin, Korepin, Slavnov 1990; Tracy, Widom 1993)

If (4) fails, not all hope is lost since

$$\det(I - K_t \restriction_{L^2(\mathbb{R}_+)}) = \exp\left[-\sum_{m=1}^{\infty} \frac{1}{m} \operatorname*{tr}_{L^2(\mathbb{R}_+)} K_t^m\right]$$

is conjugation invariant (Bertola, Cafasso 2012),

$$\begin{aligned} K_t(x,y) &= \int_0^\infty \left[\int_{\Gamma_\alpha} \hat{\phi}_t(\alpha)\mathrm{e}^{\mathrm{i}\alpha(x+z)}\mathrm{d}\alpha\right]\left[\int_{\Gamma_\beta} \hat{\psi}_t(\beta)\mathrm{e}^{-\mathrm{i}\beta(z+y)}\mathrm{d}\beta\right]\mathrm{d}z \\ &= -\mathrm{i}\int_{\Gamma_\alpha}\int_{\Gamma_\beta} \hat{\phi}_t(\alpha)\hat{\psi}_t(\beta)\mathrm{e}^{\mathrm{i}\alpha x - \mathrm{i}\beta y}\frac{\mathrm{d}\beta\,\mathrm{d}\alpha}{\alpha-\beta}, \qquad \Im(\alpha-\beta) > 0 \end{aligned}$$

and using a contour integral formula for $\chi_{(0,\infty)}(y)$ one obtains in general (e.g. for GinOE) an integrable operator in Fourier space.

Beyond integrable operators

As it happens, we can bypass the integrable shape entirely and still derive many of the operator determinants' features. In fact we will rely solely on the Hankel composition structure, but no contour integral formulæ or differential equations.

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> Hankel composition structure *unstable* under composition, resolvent not of Hankel type and *only* determinant accessible via Riemann-Hilbert problem $\rightarrow$ dynamical systems, asymptotics

Algebraic structural universality 1

Consider two Hankel operators $M_t, N_t : L^2(\mathbb{R}_+) \to L^2(\mathbb{R}_+)$

$$(M_t f)(x) := \int_0^\infty \phi_t(x+y) f(y)\,\mathrm{d}y, \ (N_t f)(x) := \int_0^\infty \psi_t(x+y) f(y)\,\mathrm{d}y$$

where $\|M_t\|_{HS}, \|N_t\|_{HS} < \infty$ for all $t \in J \subseteq \mathbb{R}$ and $\{\phi_t\}_{t\in J}, \{\psi_t\}_{t\in J}$, $\{D\phi_t\}_{t\in J}, \{D\psi_t\}_{t\in J}$ are $L^2(\mathbb{R}_+)$ dominated.

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Krajenbrink 2020, Bothner 2021

Define $K_t := M_t N_t$. If $\phi, \psi : \mathbb{R} \to \mathbb{C}$ are a.c. on $\mathbb{R}$, vanish at $+\infty$,

$$\forall\, t \in J: \quad \phi_t, \psi_t \in W^{1,2}(\mathbb{R}_+), \quad \int_0^\infty x|(D\phi_t)(x)|^2\,\mathrm{d}x < \infty,$$

then for every $t \in J$, provided $I - K_t$ is invertible on $L^2(\mathbb{R}_+)$ for all $t \in J$,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \ln F(t) = -q_0(t) q_0^*(t), \quad \begin{cases} q_0(t) := ((I-K_t)^{-1}\phi_t)(0) \\ q_0^*(t) := ((I-K_t^*)^{-1}\psi_t)(0) \end{cases}.$$

In turn, in particular for the limiting cdfs in the GOE and GinOE,

Krajenbrink 2020, Bothner 2021

Let $\epsilon > 0$. Suppose $\phi = \psi$ is continuously differentiable on $\mathbb{R}$, vanishes at $+\infty$, $\{\phi_t\}_{t\in\mathbb{R}}, \{D\phi_t\}_{t\in\mathbb{R}}$ are $L^2(\mathbb{R}_+)$ dominated

$$\forall\, t \in \mathbb{R}: \quad \phi_t \in L^1(\mathbb{R}) \cap W^{1,2}(\mathbb{R}_+), \quad \int_0^\infty x|(D\phi_t)(x)|^2 \,\mathrm{d}x < \infty,$$

and $|q_0(t)| \leq ct^{-1-\epsilon}$ for large $t > 0$.

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$$\forall\, t \in \mathbb{R}: \quad \phi_t \in L^1(\mathbb{R}) \cap W^{1,2}(\mathbb{R}_+), \quad \int_0^\infty x|(D\phi_t)(x)|^2\,\mathrm{d}x < \infty,$$

and $|q_0(t)| \leq ct^{-1-\epsilon}$ for large $t > 0$. Then, provided $I - \gamma K_t$ is invertible on $L^2(\mathbb{R}_+)$ for every $(t,\gamma) \in \mathbb{R} \times [0,1]$,

$$D(t,\gamma) = \exp\left[-\frac{1}{2}\int_t^\infty (s-t)(q_0(s;\bar{\gamma}))^2 \mathrm{d}s\right] \times$$

$$\times \sqrt{\frac{\gamma - 1 - \cosh\lambda(t;\bar{\gamma}) + \sqrt{\bar{\gamma}}\sinh\lambda(t;\bar{\gamma})}{\gamma - 2}}; \quad \lambda(t;\gamma) := \int_t^\infty q_0(s;\gamma)\,\mathrm{d}s,$$

with $q_0(t) = q_0(t;\gamma) = \sqrt{\gamma}((I - \gamma K_t)^{-1}\phi_t)(0)$.

The above results are obtained from algebraic manipulations (mostly) and they explain the universal underlying algebraic structure in our cdf formulæ for the GOE and GinOE. However, they don't tell us what

$$q_0(t) = ((I - K_t)^{-1}\phi_t)(0) \quad \text{and} \quad q_0^*(t) = ((I - K_t^*)^{-1}\psi_t)(0)$$

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Moving ahead, we build in more regularity (N-times differentiable) and integrability ($\phi_t, \psi_t \in W^{N,2}(\mathbb{R}_+)$). Then

$$q_n(t) := \big((I - K_t)^{-1}D^n\phi_t\big)(0), \qquad p_n(t) := \operatorname*{tr}_{L^2(\mathbb{R}_+)}\big((I - K_t)^{-1}D^n\phi_t \otimes \psi_t\big),$$

$$q_n^*(t) := \big((I - K_t^*)^{-1}D^n\psi_t\big)(0), \qquad p_n^*(t) := \operatorname*{tr}_{L^2(\mathbb{R}_+)}\big((I - K_t^*)^{-1}D^n\phi_t \otimes \psi_t\big),$$

defined for $t \in J$ and $n = 0, 1, \ldots, N$ satisfy the following peculiar ODE system:

$$\begin{cases} \frac{\mathrm{d}q_n}{\mathrm{d}t}(t) = q_{n+1}(t) - q_0(t)p_n(t), & \frac{\mathrm{d}p_n}{\mathrm{d}t}(t) = -q_0^*(t)q_n(t) \\ \frac{\mathrm{d}q_n^*}{\mathrm{d}t}(t) = q_{n+1}^*(t) - q_0^*(t)p_n^*(t), & \frac{\mathrm{d}p_n^*}{\mathrm{d}t}(t) = -q_0(t)q_n^*(t) \end{cases}$$

for all $n = 0, 1, \ldots, N-1$ and $t \in J$. This brings us to the analytic structural universality, first flushed out for self-adjoint Hankel composition operators by Krajenbrink in 2020.

The canonical Riemann-Hilbert problem (RHP)

Zakharov, Shabat; Ablowitz, Kaup, Newell, Segur problem

Given $t \in \mathbb{R}$ and $\phi, \psi \in L^1(\mathbb{R})$, find $\mathbf{X}(z) = \mathbf{X}(z; t, \phi, \psi) \in \mathbb{C}^{2\times 2}$ such that

(1) $\mathbf{X}(z)$ is analytic for $z \in \mathbb{C} \setminus \mathbb{R}$.

(2) $\mathbf{X}(z)$ admits continuous pointwise limits $\mathbf{X}_\pm(z) := \lim_{\epsilon\downarrow 0} \mathbf{X}(z \pm \mathrm{i}\epsilon), z \in \mathbb{R}$ which obey

$$\mathbf{X}_+(z) = \mathbf{X}_-(z) \begin{bmatrix} 1 - r_1(z) r_2(z) & -r_2(z)\mathrm{e}^{-\mathrm{i}tz} \\ r_1(z)\mathrm{e}^{\mathrm{i}tz} & 1 \end{bmatrix}, \quad z \in \mathbb{R},$$

with $r_1(z) = -\mathrm{i}\int_{-\infty}^{\infty} \phi(y)\mathrm{e}^{-\mathrm{i}zy}\,\mathrm{d}y$ and $r_2(z) = \mathrm{i}\int_{-\infty}^{\infty} \psi(y)\mathrm{e}^{\mathrm{i}zy}\,\mathrm{d}y$.

(3) Uniformly as $z \to \infty$ in $\mathbb{C} \setminus \mathbb{R}$,

$$\mathbf{X}(z) = \mathbb{I} + \mathbf{X}_1 z^{-1} + o\big(z^{-1}\big); \quad \mathbf{X}_1 = \mathbf{X}_1(t) = \big[X_1^{mn}(t)\big]_{m,n=1}^2.$$

Analytic structural universality 1

Krajenbrink 2020, Bothner 2021

Assume $\phi, \psi : \mathbb{R} \to \mathbb{C}$ are differentiable on $\mathbb{R}$, vanish at $\pm\infty$, satisfy

$$\phi, \psi \in W^{1,1}(\mathbb{R}), \quad \int_0^\infty \sqrt{\int_0^\infty |f_t(x+y)|^2 \mathrm{d}y}\, \mathrm{d}x < \infty, \quad f \in \{\phi, \psi\}.$$

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Then the above RHP is uniquely solvable provided $I - K_t$ is invertible on $L^2(\mathbb{R}_+)$. Moreover

$$\lim_{\substack{z\to\infty \\ z\notin\mathbb{R}}} z\big(\mathbf{X}(z) - \mathbb{I}\big) = \begin{bmatrix} -\mathrm{i}p_0(t) & q_0^*(t) \\ q_0(t) & \mathrm{i}p_0^*(t) \end{bmatrix}.$$

Back to our two favorite examples

With $\phi(x) = \psi(x) = \sqrt{\gamma}\,\mathrm{Ai}(x)$, we find

$$r_1(z) = \overline{r_2(z)} = -\mathrm{i}\sqrt{\gamma}\,\mathrm{e}^{\frac{\mathrm{i}}{3}z^3}, \quad z \in \mathbb{R},$$

and thus at once the standard Ablowitz-Segur Painlevé-II RHP

$$\mathbf{X}_+(z) = \mathbf{X}_-(z)\begin{bmatrix} 1-\gamma & -\mathrm{i}\sqrt{\gamma}\,\mathrm{e}^{-\frac{\mathrm{i}}{3}z^3-\mathrm{i}tz} \\ -\mathrm{i}\sqrt{\gamma}\,\mathrm{e}^{\frac{\mathrm{i}}{3}z^3+\mathrm{i}tz} & 1 \end{bmatrix}, \quad z \in \mathbb{R}.$$

With $\phi(x) = \psi(x) = \sqrt{\gamma}\,\mathrm{e}^{-x^2}/\sqrt{\pi}$, we find

$$r_1(z) = \overline{r_2(z)} = -\mathrm{i}\sqrt{\gamma}\,\mathrm{e}^{-\frac{1}{4}z^2}, \quad z \in \mathbb{R},$$

and thus at once the problem investigated in (Baik, Bothner 2018),

$$\mathbf{X}_+(z) = \mathbf{X}_-(z) \begin{bmatrix} 1 - \gamma\mathrm{e}^{-\frac{1}{2}z^2} & -\mathrm{i}\sqrt{\gamma}\,\mathrm{e}^{-\frac{1}{4}z^2 - \mathrm{i}tz} \\ -\mathrm{i}\sqrt{\gamma}\,\mathrm{e}^{-\frac{1}{4}z^2 + \mathrm{i}tz} & 1 \end{bmatrix}, \quad z \in \mathbb{R}.$$

There is more to our story

Consider the Laguerre Orthogonal Ensemble (LOE), i.e. matrices

$$\mathbf{X} = \mathbf{Y}^\top \mathbf{Y} \in \mathbb{R}^{n\times n}: \quad \mathbf{Y} \in \mathbb{R}^{m\times n},\ Y_{ij} \overset{\text{iid}}{\sim} N(0,1),\ m \geq n.$$ (Wishart 1928)

It is known that, as $n, m \to \infty$ with $\frac{n}{m} \to 1$,

$$\min_{i=1,\ldots,n} \lambda_i(\mathbf{X}) \Rightarrow \frac{F_\alpha}{4n},$$ (Bronk 1965; Marchenko, Pastur 1967; Forrester 1993)

Forrester 2000

$$\mathbb{P}(F_\alpha \geq t) = \exp\left[-\frac{1}{8}\int_0^t \ln\left(\frac{t}{s}\right)(q_\alpha(s))^2\,\mathrm{d}s - \frac{1}{4}\int_0^t q_\alpha(s)\,\frac{\mathrm{d}s}{\sqrt{s}}\right] \qquad (5)$$

and $q_\alpha = q_\alpha(s)$ solves an ODE boundary value problem

The limiting cdf (5) (LOE) looks suspiciously similar to other limiting cdfs in real hard edge RMT ensembles, at least on the structural surface (e.g. product ensembles, Muttalib-Borodin ensembles, chain ensembles). Perhaps a coincidence?

Probabilistic Universality

The above limit law is universal in the class of sample covariance matrices (Soshnikov 2002; Péché 2009; Tao, Vu 2012).

Why so similar?

In obtaining the above exact formulæ, one typically

(1) starts from the model's finite n correlation functions,

(2) computes a finite n gap probability as operator determinant,

(3) then passes to a suitable large n limit.

For LOE the *limiting* distribution function equals

$$D(t,1) \text{ “} = \text{''} \sqrt{\det(I - \bar{\gamma}K_t - \gamma\phi_t \otimes \varphi_t \restriction_{L^2(0,1)})}\,\Big|_{\gamma=1},$$

where K_t is a Hankel composition operator with kernel

$$K_t(x,y) = t\int_0^1 \phi_t(xz)\psi_t(zy)\,\mathrm{d}z; \quad f_t(x) := f(xt). \tag{6}$$

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Precisely, $\phi(x) = \psi(x) = \frac{1}{2}J_\alpha(\sqrt{x})$.

Once the structure (6) has been flushed out one attempts to massage it in integrable shape, i.e. one tries to find $f_j, g_j \in L^\infty(\mathbb{R}_+)$ such that

$$\frac{\sum_{j=1}^N f_j(x)g_j(y)}{x-y} = K_t(x,y) = t\int_0^1 \phi_t(xz)\psi_t(zy)\,\mathrm{d}z. \tag{7}$$

This can (e.g. LOE) or cannot (e.g. product ensembles, Muttalib-Borodin ensembles, chain ensembles) work out, see (Blower 2008), but is considered in general desirable given that integrable operators share many remarkable properties:

stable under composition, resolvent of same type and accessible via Riemann-Hilbert problem $\rightarrow$ dynamical systems, asymptotics (Its, Izergin, Korepin, Slavnov 1990; Tracy, Widom 1993)

If (7) fails, not all hope is lost since

$$\det(I - K_t \restriction_{L^2(0,1)}) = \exp\left[-\sum_{m=1}^{\infty} \frac{1}{m} \operatorname*{tr}_{L^2(0,1)} K_t^m\right]$$

is conjugation invariant (Girotti 2014),

$$\begin{aligned} K_t(x,y) &= t\int_0^1 \left[\int_{\Gamma_\alpha} \hat{\phi}_t(\alpha)(xz)^{-\alpha}\mathrm{d}\alpha\right]\left[\int_{\Gamma_\beta} \hat{\psi}_t(\beta)(zy)^{\beta-1}\mathrm{d}\beta\right]\mathrm{d}z \\ &= -t\int_{\Gamma_\alpha}\int_{\Gamma_\beta} \hat{\phi}_t(\alpha)\hat{\psi}_t(\beta)x^{-\alpha}y^{\beta-1}\frac{\mathrm{d}\beta\,\mathrm{d}\alpha}{\alpha-\beta}, \qquad \Re(\beta-\alpha) > 0 \end{aligned}$$

and using a contour integral formula for $\chi_{(0,1)}(y)$ one obtains in general (e.g. for GinOE) an integrable operator in Mellin space.

Beyond integrable operators

As it happens, we can bypass the integrable shape entirely and still derive many of the operator determinants' features. In fact we will rely solely on the Hankel composition structure, but no contour integral formulæ or differential equations. However there is a price to pay:

> Hankel composition structure *unstable* under composition, resolvent not of Hankel type and *only* determinant accessible via Riemann-Hilbert problem $\rightarrow$ dynamical systems, asymptotics

Algebraic structural universality 2

Consider two Hilbert-Schmidt Hankel operators $M_t, N_t : L^2(0,1) \to L^2(0,1)$,

$$(M_t f)(x) := \sqrt{t}\int_0^1 \phi_t(xy)f(y)\,\mathrm{d}y,\ \ (N_t f)(x) := \sqrt{t}\int_0^1 \psi_t(xy)f(y)\,\mathrm{d}y$$

where $t \in J \subseteq \mathbb{R}_+$ and $\{\phi_t\}_{t\in J}, \{\psi_t\}_{t\in J}, \{D\phi_t\}_{t\in J}$, $\{D\psi_t\}_{t\in J}$ are $L^2(0,1)$ dom.

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Bothner 2021

Define $K_t := M_t N_t$. If $\phi, \psi : \mathbb{R}_+ \to \mathbb{C}$ are a.c. on $\mathbb{R}_+$, are $o(x^{-1/2})$ near zero,

$$\forall\, t \in J: \quad \phi_t, \psi_t \in H^{1,2}(0,1), \qquad \int_0^1 |(MD\phi_t)(x)|^2 \ln x \,\mathrm{d}x < \infty,$$

then for every $t \in J$, provided $I - K_t$ is invertible on $L^2(0,1)$ for all $t \in J$,

$$t\frac{\mathrm{d}}{\mathrm{d}t}\left[t\frac{\mathrm{d}}{\mathrm{d}t}\ln F(t)\right] = -q_0(t)q_0^*(t), \quad \begin{cases} q_0(t) := ((I-K_t)^{-1}\phi_t)(1) \\ q_0^*(t) := t((I-K_t^*)^{-1}\psi_t)(1) \end{cases}.$$

In turn, in particular for the limiting cdf in the LOE,

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Let $\epsilon > 0$. Suppose $\phi = \psi$ is continuously differentiable on $\mathbb{R}_+$, is $o(x^{-1/2})$ near zero, $\{\phi_t\}_{t\in\mathbb{R}}, \{MD\phi_t\}_{t\in\mathbb{R}}$ are $L^2(0,1)$ dominated

$$\forall\, t \in \mathbb{R}_+ : \quad \phi_t \in L^1_\circ(\mathbb{R}_+) \cap H^{1,2}(0,1), \quad \int_0^\infty |(MD\phi_t)(x)|^2 \ln x \, \mathrm{d}x < \infty,$$

and $|q_0(t)| \leq ct^{-\frac{1}{2}+\epsilon}$ for small $t > 0$. Then, provided $I - \gamma K_t$ is invertible on $L^2(0,1)$ for every $(t,\gamma) \in \mathbb{R} \times [0,1]$,

$$D(t,\gamma) = \exp\left[-\frac{1}{2}\int_0^t \ln\left(\frac{t}{s}\right)(q_0(s;\bar{\gamma}))^2 \mathrm{d}s\right] \times$$

$$\times \sqrt{\frac{\gamma - 1 - \cosh\lambda(t;\bar{\gamma}) + \sqrt{\bar{\gamma}}\sinh\lambda(t;\bar{\gamma})}{\gamma - 2}}; \quad \lambda(t;\gamma) := \int_0^t q_0(s;\gamma)\,\frac{\mathrm{d}s}{\sqrt{s}},$$

with $q_0(t) = q_0(t;\gamma) = \sqrt{\gamma}((I - \gamma K_t)^{-1}\phi_t)(1)$.

The above results are obtained from algebraic manipulations (mostly) and they explain the universal underlying algebraic structure in our cdf formula for the LOE. However, they don't tell us what

$$q_0(t)=\big((I-K_t)^{-1}\phi_t\big)(1)\quad\text{and}\quad q_0^*(t)=t\big((I-K_t^*)^{-1}\psi_t\big)(1)$$

are.

Moving ahead, we build in more regularity (N-times differentiable) and integrability ($\phi_t,\psi_t\in H^{N,2}(0,1)$). Then

$$q_n(t):=\big((I-K_t)^{-1}(MD)^n\phi_t\big)(1),\qquad p_n(t):=t\operatorname*{tr}_{L^2(0,1)}\big((I-K_t)^{-1}(MD)^n\phi_t\otimes\psi_t\big),$$

$$q_n^*(t):=t\big((I-K_t^*)^{-1}(DM)^n\psi_t\big)(1),\quad p_n^*(t):=t\operatorname*{tr}_{L^2(0,1)}\big((I-K_t^*)^{-1}(DM)^n\psi_t\otimes\phi_t\big)$$

defined for $t\in J$ and $n=0,1,\ldots,N$ satisfy the following peculiar ODE system:

$$\begin{cases} t\frac{\mathrm{d}q_n}{\mathrm{d}t}(t) = q_{n+1}(t) + q_0(t)p_n(t), & t\frac{\mathrm{d}p_n}{\mathrm{d}t}(t) = q_0^*(t)q_n(t) \\ t\frac{\mathrm{d}q_n^*}{\mathrm{d}t}(t) = q_{n+1}^*(t) + q_0^*(t)p_n^*(t), & t\frac{\mathrm{d}p_n^*}{\mathrm{d}t}(t) = q_0(t)q_n^*(t) \end{cases}$$

for all $n = 0, 1, \ldots, N-1$ and $t \in J$. This brings us to the analytic structural universality.

The canonical Riemann-Hilbert problem (RHP)

Zakharov, Shabat; Ablowitz, Kaup, Newell, Segur problem

Given $t \in \mathbb{R}_+$ and $\phi, \psi \in L^1_\circ(\mathbb{R}_+)$, find $\mathbf{X}(z) = \mathbf{X}(z; t, \phi, \psi) \in \mathbb{C}^{2\times 2}$ such that

(1) $\mathbf{X}(z)$ is analytic for $z \in \mathbb{C} \setminus (\frac{1}{2} + \mathrm{i}\mathbb{R})$.

(2) $\mathbf{X}(z)$ admits continuous pointwise limits $\mathbf{X}_\pm(z) := \lim_{\epsilon \downarrow 0} \mathbf{X}(\frac{1}{2} \mp \epsilon + \mathrm{i}z)$, $z \in \mathbb{R}$ which obey

$$\mathbf{X}_+(z) = \mathbf{X}_-(z) \begin{bmatrix} 1 - r_1(z) r_2(z) & -r_2(z) t^z \\ r_1(z) t^{-z} & 1 \end{bmatrix}, \quad z \in \frac{1}{2} + \mathrm{i}\mathbb{R},$$

with $r_1(z) = \int_0^\infty \phi(y) y^{z-1} \, \mathrm{d}y$ and $r_2(z) = \int_0^\infty \psi(y) y^{-z} \, \mathrm{d}y$.

(3) Uniformly as $z \to \infty$ in $\mathbb{C} \setminus (\frac{1}{2} + \mathrm{i}\mathbb{R})$,

$$\mathbf{X}(z) = \mathbb{I} + \mathbf{X}_1 z^{-1} + o\big(z^{-1}\big); \quad \mathbf{X}_1 = \mathbf{X}_1(t) = \big[X_1^{mn}(t)\big]_{m,n=1}^2.$$

Analytic structural universality 2

Bothner 2021

Assume $\phi, \psi : \mathbb{R}_+ \to \mathbb{C}$ are differentiable on $\mathbb{R}_+$ with $\sqrt{x}f(x)$ bounded near zero and infinity for $f \in \{\phi, \psi\}$ and

$$\forall\, t \in \mathbb{R}_+ : \quad \phi_t, \psi_t \in H_\circ^{1,1}(\mathbb{R}_+), \quad \int_0^1 \sqrt{\int_0^1 |f_t(xy)|^2 \mathrm{d}y}\, \frac{\mathrm{d}x}{\sqrt{x}} < \infty, \quad f \in \{\phi, \psi\}.$$

Then the above RHP is uniquely solvable provided $I - K_t$ is invertible on $L^2(0,1)$. Moreover

$$\lim_{\substack{z\to\infty \\ z\notin \frac{1}{2}+\mathrm{i}\mathbb{R}}} z\big(\mathbf{X}(z) - \mathbb{I}\big) = \begin{bmatrix} p_0(t) & q_0^*(t) \\ -q_0(t) & -p_0^*(t) \end{bmatrix}.$$

One hard edge example

Take $\phi(x) = \psi(x) = \frac{1}{2}\sqrt{\gamma} J_\alpha(\sqrt{x})$, then for $z \in \frac{1}{2} + \mathrm{i}\mathbb{R}$,

$$r_1(z) = \sqrt{\gamma}\, 2^{2z-1} \frac{\Gamma(\frac{\alpha}{2} + z)}{\Gamma(\frac{\alpha}{2} - z + 1)}, \quad r_2(z) = \sqrt{\gamma}\, 2^{1-2z} \frac{\Gamma(\frac{\alpha}{2} - z + 1)}{\Gamma(\frac{\alpha}{2} + z)}$$

and thus at once

$$\mathbf{X}_+(z) = \mathbf{X}_-(z) \begin{bmatrix} 1-\gamma & -r_2(z)t^z \\ r_1(z)t^{-z} & 1 \end{bmatrix}, \; z \in \frac{1}{2} + \mathrm{i}\mathbb{R},$$

which yields the well-known Painlevé-V connection.

Going further

More general than ordinary additive or multiplicative Hankel operators, we can also analyze weighted Hankel operators.

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Here are some details for additive weighted Hankel operators. Let $w : \mathbb{R} \to \mathbb{R}_{\geq 0}$ denote a differentiable, nondecreasing and bounded function on $\mathbb{R}$ such that $\int_{-\infty}^{0} w(x)\,\mathrm{d}x < \infty$.

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Here are some details for additive weighted Hankel operators. Let $w:\mathbb{R}\to\mathbb{R}_{\geq 0}$ denote a differentiable, nondecreasing and bounded function on $\mathbb{R}$ such that $\int_{-\infty}^{0} w(x)\,\mathrm{d}x<\infty$.

Consider $M_t: L^2(\mathbb{R})\to L^2(\mathbb{R}_+)$ and $N_t: L^2(\mathbb{R}_+)\to L^2(\mathbb{R})$

$$(M_tf)(x):=\int_{-\infty}^{\infty}\phi_t(x+y)\sqrt{w(y)}f(y)\,\mathrm{d}y,$$

$$(N_tf)(x):=\int_{0}^{\infty}\sqrt{w(x)}\psi_t(x+y)f(y)\,\mathrm{d}y,$$

assuming $\|M_t\|_{HS}<\infty$ and $\|N_t\|_{HS}<\infty$ for all $t\in J\subseteq\mathbb{R}$.

Algebraic structural universality 3

Bothner 2021

Define $K_t := M_t N_t$. If $\phi, \psi : \mathbb{R} \to \mathbb{C}$ are a.c. on $\mathbb{R}_+$, vanish at $\pm\infty$,

$$\forall\, t \in J: \quad \int_0^\infty \left[\int_{-\infty}^\infty |(D\phi_t)(x+y)|^2 w(y)\,\mathrm{d}y\right]\mathrm{d}x < \infty,$$

the families $\{\phi_t\}_{t\in J}, \{\psi_t\}_{t\in J}, \{D\phi_t\}_{t\in J}, \{D\psi_t\}_{t\in J}$ are $L^2_w(\mathbb{R})$ dominated and with $\mathrm{d}\nu(z) := w'(z)\mathrm{d}z$, for every $s \in \mathbb{R}$

$$\phi_s, \psi_s \in W^{1,2}_\nu(\mathbb{R}_+) := \Big\{ f \in W^{1,2}(\mathbb{R}_+) : \int_{-\infty}^\infty \|f_z\|^2_{L^2(\mathbb{R}_+)}\mathrm{d}\nu(z) < \infty, \int_{-\infty}^\infty \|Df_z\|^2_{L^2(\mathbb{R}_+)}\mathrm{d}\nu(z) < \infty \Big\}.$$

Then for every $t \in J$, provided $I - K_t$ is invertible on $L^2(\mathbb{R}_+)$ for all $t \in J$,

$$\frac{\mathrm{d}^2}{\mathrm{d}t^2} \ln F(t) = -\int_{-\infty}^\infty q_0(t,z) q_0^*(t,z)\mathrm{d}\nu(z), \quad \begin{cases} q_0(t,z) := ((I-K_t)^{-1}\phi_{t+z})(0) \\ q_0^*(t,z) := ((I-K_t^*)^{-1}\psi_{t+z})(0) \end{cases}.$$

Analytic structural universality 3

Also in the weighted setup we can characterize the Fredholm determinant $F(t)$ through a canonical RHP, however the problem is operator-valued, i.e. we don't seek a matrix $\mathbf{X}(z) \in \mathbb{C}^{2\times 2}$ with prescribed analytic and asymptotic properties, but instead an (integral) operator $\mathbf{X}(z) \in \mathcal{L}(\mathcal{H} \oplus \mathcal{H})$.

Analytic structural universality 3

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This is more technical than the previous setup, still it allows us to systematically study Fredholm determinants, for one example see the recent (Bothner, Cafasso, Tarricone 2021).

Integro-differential equations

Consider the complex elliptic Ginibre Ensemble (eGinUE), i.e. matrices

$$\mathbf{X} = \mathbf{Y}_1 + \mathrm{i}\sqrt{\frac{1-\tau}{1+\tau}}\,\mathbf{Y}_2 \in \mathbb{C}^{n\times n}: \quad \mathbf{Y}_k \overset{\text{iid}}{\sim} \text{GUE},\ \tau \in [0,1]. \qquad \text{(Girko 1985)}$$

It is known that, as $n \to \infty, \tau \uparrow 1 : n^{1/6}\sqrt{1-\tau} \to \sigma \geq 0$,

$$\max_{i=1,\ldots,n} \Re\lambda_i(\mathbf{X}) \Rightarrow c_{n,\tau,\sigma} + \frac{F_\sigma}{a_{n,\tau,\sigma}}, \qquad \text{(Bender 2010)}$$

where $\mathbb{P}(F_\sigma \leq t) = \det(I - M_\sigma \restriction_{L^2((t,\infty)\times\mathbb{R})})$ is determined through

$$M_\sigma(z_1,z_2)=\frac{\mathrm{i}}{4\pi^{5/2}}\int_\gamma\int_\gamma\frac{\mathrm{e}^{\mathrm{i}(\frac{1}{3}\lambda^3+x_1\lambda)}\mathrm{e}^{\mathrm{i}(\frac{\mu^3}{3}+x_2\mu)}}{\lambda+\mu}$$
$$\times\,\mathrm{e}^{-\frac{1}{2}(\sigma\lambda+y_1)^2}\mathrm{e}^{-\frac{1}{2}(\sigma\mu-y_2)^2}\,\mathrm{d}\lambda\,\mathrm{d}\mu;\qquad z_k=(x_k,y_k)\in\mathbb{R}^2.$$

As it happens, the Hankel method works in this "higher-dimensional" problem

Bothner, Little 2021

$$\mathbb{P}(F_\sigma\leq t)=\det(I-K_\sigma\restriction_{L^2((t,\infty)\times\mathbb{R})}),\quad (t,\sigma)\in\mathbb{R}\times[0,\infty),$$
$$K_\sigma(z_1,z_2)=\frac{1}{\sqrt{\pi}}\mathrm{e}^{-\frac{1}{2}y_1^2}K_{\mathrm{Ai}}(x_1+\sigma y_1,x_2+\sigma y_2)\mathrm{e}^{-\frac{1}{2}y_2^2},$$

and so $\mathbb{P}(F_\sigma\leq t)$ is expressible in terms of an integro-differential Painlevé-II transcendent.

Thank you very much for your attention!!!

There is another

What about truncated/finite Wiener-Hopf operators, i.e.

$$W_t : L^2(0,1) \to L^2(0,1) \qquad (W_t f)(x) := t \int_0^1 \eta\big(t(x-y)\big) f(y)\,\mathrm{d}y$$

with $t \in \mathbb{R}_+$ and where

$$\eta(x) = \frac{1}{2\pi} \int_{-\infty}^{\infty} (\sigma(y) - 1) \mathrm{e}^{-\mathrm{i}xy}\,\mathrm{d}y, \quad \sigma - 1 \in L^1(\mathbb{R}).$$

Using the continuous version of the Borodin-Okounkov identity also these determinants can be analyzed in our framework:

Basor, Chen 2003

We have for every $t \in \mathbb{R}_+$

$$\det(I - W_t \restriction_{L^2(0,1)}) = Ze^{ct} \det(I - K_t \restriction_{L^2(\mathbb{R}_+)})$$

where $c = \int_{-\infty}^{\infty} \ln \sigma(y)\, \mathrm{d}y$ and K_t is an additive Hankel composition operator whose kernel is constructed in terms of the Wiener-Hopf factors associated with σ (implicitly assuming that the Wiener-Hopf factorization of σ uniquely exists).

Note that the standard sine kernel

$$\eta(x) = \frac{\sin x}{\pi x}$$

is *not* of Wiener-Hopf type. One must use a different approach.