Multiple $SLE_\kappa$ from a loop measure perspective

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Goal: combine dynamical and configurational interpretations of SLE to understand multiple radial SLE (two kinds)

Today’s talk:

Many authors:
- Dubédat ’07
- Kozdron, Lawler [’07]
- Kytölä and Peltola [’16].
- Jahangoshahi, Lawler [’18]
- Peltola, Wu [’19]
- Zhan [’18, ’19]
- Beffara, Peltola, Wu [’21]

Notes:
- Loop measure will be a main tool
- Discrete models are motivation
- This talk: $\kappa \leq 4$
Main Result

Chordal case: Known that multiple $SLE$ is absolutely continuous w.r.t. $n$ independent $SLE$s (boundary perturbation property)

Obstacle for radial: want to do an analogous construction, but
(Radon-Nikodym derivative $= \infty$) $\times$ (partition function $= 0$) $= ??$

Short answer:
- Tilt independent $n$-radial $SLE$ $\rightarrow$ locally independent $SLE$
- Tilt again and take a limit $\rightarrow$ global $n$-radial $SLE$

Main tools: Brownian loop measure, analysis of radial Bessel process

Main Result: Radial Bessel/Dyson BM naturally appears as the driving function!
(Different drift for locally independent vs. global)
Contents

I. Background
  • Loewner Equation & SLE
    • Interpretations: dynamical vs configurational
  • Loop measures
    • Loop-erased random walk
    • Restriction property and RW loop measure
  • Radial restriction property & Brownian loop measure

II. $n$-Radial SLE
  • Which loops?
  • Locally Independent $n$-radial SLE, connection to radial Bessel
  • (Global) $n$-radial SLE, connection to radial Bessel
I. Background

Chordal Loewner Equation

\( \gamma : (0,T] \to \mathbb{H} \) simple curve, \( \gamma(0) \in \mathbb{R} \).

Composition property: \( g_{s,t} \circ g_{s}(z) = g_{t}(z) \).

Loewner (1920s): \( g_t \) satisfies

\[
\dot{g}_t(z) = \frac{\dot{b}(t)}{g_t(z) - U(t)}, \quad g_0(z) = z.
\]
Radial Loewner Equation (one curve)

Radial Loewner equation: (from a boundary point to 0)

- $\gamma : (0,T] \to \mathbb{D}$ a simple curve starting on unit circle.

- Conformal mappings $g_t : \mathbb{D} \setminus \gamma_t \to \mathbb{D}$ satisfy

$$\dot{g}_t(w) = 2a \ g_t(w) \frac{z_t + g_t(w)}{z_t - g_t(w)}.$$

- Parameterized so that $g'_t(0) = e^{2at}$. 
Schramm-Loewner Evolution

Dynamical interpretation

• Chordal:
  \[ \dot{g}_t(z) = \frac{2a}{g_t(z) - B_t}, \quad g_0(z) = z. \]

• Radial:
  \[ \dot{g}_t(w) = 2a \, g_t(w) \frac{e^{2iB_t} + g_t(w)}{e^{2iB_t} - g_t(w)}, \quad g_0(w) = w. \]

Results usually stated in terms of \( \kappa \), where \( a = 2/\kappa \).
Schramm-Loewner Evolution

Configurational interpretation

- Conformal invariance/covariance
  - Idea: $f(\text{SLE in } D) = \text{SLE in } f(D)$
  - Measures with total mass:
    \[
    f \circ \mu_D(z,0) = |f'(z)|^b |f'(0)|^{\tilde{b}} \mu_{f(D)}(f(z),0) .
    \]
    \[SLE \text{ measure on curves from } z \text{ to } 0\]
  - Partition functions (i.e. total mass)
    \[
    \Psi_D(z,0) = |f'(z)|^b |f'(0)|^{\tilde{b}} \Psi_{f(D)}(f(z),0) .
    \]
- Domain Markov property
  - Idea: curve views its own past as part of the boundary

Non-probability measures: keep more information as we change the domain

\[
\begin{align*}
  b &= \frac{6 - \kappa}{2 \kappa} \\
  \tilde{b} &= \frac{b(1 - a)}{2a} \\
  a &= 2 / \kappa
\end{align*}
\]
Schramm-Loewner Evolution

- Universal scaling limit of many discrete processes, including:
  - Loop-erased random walk
  - Critical Ising model
  - Uniform spanning tree
  - Critical percolation

Today: use discrete models to build intuition

We’ll look at loop-erased random walk to understand loop measures
Loop measure and LERW

Loop-erased random walk:

How does the measure of $\gamma$ in $D$ compare to its measure in $D\setminus K$?
Loop measure and LERW

Loop-erased random walk:

How does the measure of \( \gamma \) in \( D \) compare to its measure in \( D \setminus K \)?

LERW path carries measure of SRW loops that intersect \( \gamma \) and \( K \).

Need to reweight the measure by \( m\{ \ell \in D : \ell \cap K \neq \emptyset \} \).
Random Walk Loop Measure in $\mathbb{Z}^2$

- **(Unrooted) loop measure:**

$$m(\ell) = \frac{K(\ell)}{|\ell| \cdot 4|\ell|}.$$  

- **Why this def?** **Limit = Brownian loop measure.** (Want SLE results.) [Lawler-Werner-Trujillo Ferreras]

- **Brownian loop measure** $m_\mathbb{C}$ on unrooted loops given by 

$$(\text{duration 1 Brownian bridge}) \times (\text{Area meas.}) \times \left( \frac{1}{2\pi t^2} dt \right)$$

- **Base loop**
- **Basepoint (i.e. root)**
- **time duration $\times$ root location**

$K(\ell) = \# \text{ of representatives of } \ell$
**Restriction**

<table>
<thead>
<tr>
<th>Measure on paths in $D$</th>
<th>Measure on paths in $D \setminus K$</th>
</tr>
</thead>
<tbody>
<tr>
<td>• LERW</td>
<td>• weight by $m{\ell \in D : \ell \cap K \neq \emptyset}$</td>
</tr>
<tr>
<td>• $SLE_\kappa$</td>
<td>• weight by $m{\ell \in D : \ell \cap K \neq \emptyset}$</td>
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- Want to study: $SLE$ paths weighted by **Brownian loop measure**
- Need stochastic calculus to make sense of this
Girsanov Theorem (stoch. calc.)

**Girsanov Theorem:** (giving drift to $B_t$ via change of measure)

- $B_t$ Brownian motion under probability measure $\mathbb{P}$.
- $M_t$ a non-negative martingale wrt $\mathbb{P}$, $M_0 = 1$,
  \[ dM_t = A_t M_t dB_t. \]
- Let \( \frac{d\tilde{\mathbb{P}}}{d\mathbb{P}} = M_t \).
- Then $B_t$ satisfies
  \[ dB_t = A_t dt + dW_t, \]
  for $W_t$ Brownian motion wrt $\tilde{\mathbb{P}}$.

**Application to SLE:**

- $SLE_\kappa$ has driving function $\sqrt{\kappa B_t}$.
- new measure $\rightarrow$ new driving function.
Restriction property for radial $SLE_\kappa$

- $\gamma$ radial $SLE_\kappa$ from 1 to 0 in $\mathbb{D}$. $\gamma_t = \gamma[0,t]$.
- $U = \mathbb{D}\setminus K \subset \mathbb{D}$ simply connected
- Let $D_t = \mathbb{D}\setminus \gamma_t$, $U_t = U\setminus \gamma_t$, 
  \[ \Psi_t = \frac{\Psi_{U_t}(\gamma(t),0)}{\Psi_{D_t}(\gamma(t),0)}. \]

- $SLE_\kappa$ in $U$ is $SLE_\kappa$ in $\mathbb{D}$ “weighted locally” by $\Psi_t$
  - Find a local martingale $M_t = A_t \Psi_t$, where $A_t$ is differentiable.
  - Use Girsanov theorem.
  - Can calculate: $A_t = 1\{\gamma_t \subset U\} \exp\left\{ \frac{c}{2} m_D(\gamma_t,K) \right\}$. Proof: Jahangoshahi & Lawler ’18, earlier folklore?

Initial segment
Total mass of paths in $U_t$
Total mass of paths in $D_t$
II. n-Radial SLE

Multiple Radial $SLE_\kappa$

- **Radial Loewner equation**: (from a boundary point to 0)
  \[
  \dot{g}_t(z) = 2a \cdot g_t(z) \sum_{j=1}^{n} \frac{z_j^j + g_t(z)}{z_j^j - g_t(z)}
  \]

- $SLE = a$ measure on paths (with partition function).

- $SLE$ also = **a measure on parametrized curves with killing**.
  - Curves are growing: keep time parameter $t$
  - Process up to a stopping time $t < T$.
  - Paths killed, so not a probability measure:
  - Total mass at time $t = \text{partition function } \Psi_{\mathbb{D}\setminus\gamma_t}(\gamma(t),0)$.

- Can use both “configurational” and “dynamical” information!
Multiple Radial $SLE_\kappa$

**Q:** Use restriction to define multiple radial $SLE$?

- $n$ curves in the disk from unit circle to origin.
- Grow all curves at the “same rate.”
  - Measure on parametrized curves, not just paths.
  - Weight by loops that hit multiple curves
- Procedure works for chordal case

[Jahangoshahi & Lawler ’18]
Multiple Radial $SLE_{\kappa}$

**Q:** Use restriction to define multiple radial $SLE$?

- $n$ curves in the disk from unit circle to origin.
- Grow all curves at the “same rate.”
  - Measure on parametrized curves, not just paths.
  - Weight by loops that hit multiple curves
- Procedure works for chordal case
  - [Jahangoshahi & Lawler '18]
- But in radial case, this measure is infinite! (tiny red loops)
- We’ll need to define it as a limit process.
**SLE\(_{\kappa}\) Results** [H-Lawler, ’21]

Setup

- \(n\) points on the unit circle:
  \[ z^j_t = \exp\{2i\theta^j_t\}, \quad j = 1, \ldots, n. \]

- Define \(h\) by:
  \[ g_t(e^{2i\zeta}) = e^{2ih_t(\zeta)}. \]

Then

\[ \dot{h}_t(\zeta) = a \sum_{j=1}^{n} \cot(h_t(\zeta) - \theta^j_t). \]

- Killing: \(0 \leq t \leq T\) stop process when any \(\gamma^j_t \cap \gamma^k_t \neq \emptyset\).
Loop notation setup

- Denote the $n$-tuple of curves $\gamma_t = (\gamma_t^1, \ldots, \gamma_t^n)$
- Let $\hat{L}_t^j = \hat{L}_t^j(\gamma)$ be the set of loops that hit $\gamma_t^j$ after hitting another $\gamma_t^k$.
- Define
  \[
  \hat{\mathcal{L}}_t = \hat{I}_t \exp \left\{ \frac{c}{2} \sum_{j=1}^{n} m_\mathbb{D}(\hat{L}_t^j) \right\}.
  \]
Which loops?

**Locally independent SLE**

- Loops used for $\hat{\mathcal{L}}_t$

- At time $t$, each curve $\gamma_i^j$ sees the past of every other curve

- $t$-measurable loops

**Truncation whose limit is global $n$-radial SLE**

- Loops used for $\mathcal{L}_t$

- **Future** loops also included

- Truncate by assuming one hit is before time $t$. 
Locally Independent $SLE_\kappa$

“Theorem” (Locally independent $SLE_\kappa$) [H-Lawler ’21]: If each of the $n$ SLE curves grows as if in the domain $D \setminus \gamma_t$, then the driving functions satisfy

$$d\theta^j_t = a \sum_{k:k \neq j} \cot(\theta^j_t - \theta^k_t) \, dt + dW^j_t,$$

for $W^1_t, \ldots, W^n_t$ independent Brownian motions.

Radial Bessel equation with parameter $a = 2/\kappa$ generates locally independent $SLE_\kappa$. 
**Locally Independent \( SLE_\kappa \)**

**Theorem (Locally independent \( SLE_\kappa \))** [H-Lawler ’21]:

If \( \gamma_i \sim \text{independent} \ n\text{-path} \ SLE_{\kappa}, \) then

\[
M_t = \mathcal{L}_t \Psi_t \exp \left\{ -2\tilde{a}bn(n - 1)t + ab \int_0^t \phi(\theta_s)ds \right\}
\]

is a local martingale for \( 0 \leq t \leq T. \)

If \( \mathbb{P}_* \) is defined by

\[
\frac{d\mathbb{P}_*}{d\mathbb{P}} = M_t,
\]

then

\[
d\theta_t^j = a \sum_{k:k \neq j} \cot(\theta_t^j - \theta_t^k) \, dt + dW_t^j,
\]

for \( W_t^1, \ldots, W_t^n \) independent Brownian motions under \( \mathbb{P}_* \).

Radial Bessel equation with parameter \( a = 2/\kappa \)
generates locally independent \( SLE_\kappa. \)
Idea of Proof

Can express $M_t = \prod_{j=1}^{n} M_t^j$

- $M_t^j$ is just the “j-version” of every term in $M_t$:
  
  $$M_t^j = I_t^j \Psi_t^j \exp \left\{ \frac{c}{2} m_{\mathbb{D}}(\hat{L}_t^j) \right\} \exp \left\{ ab \int_0^t \sum_{k \neq j} \csc^2(\theta_s^j - \theta_s^k)ds \right\}$$

- After tilting by $M_t^j$, the curve $\gamma^j$ at time $t$ is locally growing as SLE in $D_t = \mathbb{D} \setminus \gamma_t$
  
  - Reason for the name locally independent SLE

- Computation verifies local mart; Girsanov finishes the proof.
Locally Independent $SLE_\kappa$

Locally independent $SLE_\kappa$: In the measure $\mathbb{P}_*$, the paths $\gamma^1_t, \ldots, \gamma^n_t$ locally grow like independent $SLE$ in slit domain.

- Driving functions satisfy
  \[
  d\theta^j_t = a \sum_{k: k \neq j} \cot(\theta^j_t - \theta^k_t) \, dt + dW^j_t.
  \]
- Drift strong enough to guarantee non-intersection of driving functions, but paths could collide.
- $\hat{\mathcal{L}}_t$ doesn’t see any loops that hit in the future.
- Idea: $\gamma^j_t$ only “planning” to avoid past of other curves
- Need to tilt again!
Global \(n\)-radial \(SLE_\kappa\)

**Theorem (Global \(n\)-radial \(SLE_\kappa\)) [H-Lawler ’21]:**
Let \(t\) be fixed, and let \(\gamma_t \sim \text{independent } n\text{-path } SLE_\kappa\).
For \(T > t\), let \(\mu_T = \mu_{T,t}\) denote the measure on \(\gamma_t\) whose Radon-Nikodym derivative with respect to \(\mathbb{P}\) is

\[
\frac{\mathcal{L}_T}{\mathbb{E}^{\theta_0} [\mathcal{L}_T]}.
\]

Then as \(T \to \infty\), \(\mu_T\) converges in variation distance to a prob measure under which the driving functions satisfy

\[
d\theta_t^j = 2a \sum_{k: k \neq j} \cot(\theta_t^j - \theta_t^k) \, dt + d\hat{W}_t^j,
\]

for \(\hat{W}_t^1, \ldots, \hat{W}_t^n\) independent Brownian motions.

Radial Bessel equation with parameter \(2a = 4/\kappa\)
generates \(n\)-radial \(SLE_\kappa\).
Idea of Proof

Radial Bessel calculations:

- Define martingale
  \[ M_{t,\alpha} = \prod_{1 \leq j < k \leq n} |\sin(\theta^k - \theta^j)|^\alpha \exp \left\{ \frac{\alpha^2 n(n^2 - 1)}{6} t \right\} \exp \left\{ \frac{\alpha - \alpha^2}{2} \int_0^t \psi(\theta_s) ds \right\} \]
- Tilting by \( M_{t,\alpha} \) gives a prob measure \( \mathbb{P}_\alpha \) with
  \[ d\theta^j_t = \alpha \sum_{k: k \neq j} \cot(\theta^j_t - \theta^k_t) dt + dW^j_t. \]
  Locally indep SLE if \( \alpha = a \)

- Define
  \[ N_{t,\alpha,2\alpha} = \prod_{1 \leq j < k \leq n} |\sin(\theta^k - \theta^j)|^\alpha \exp \left\{ \frac{\alpha^2 n(n^2 - 1)}{2} t \right\} \exp \left\{ -\frac{\alpha(3\alpha - 1)}{2} \int_0^t \psi(\theta_s) ds \right\} \]
- This is a \( \mathbb{P}_\alpha \)-martingale.
- Tilting by \( M_{t,\alpha} \) and then by \( N_{t,\alpha,2\alpha} \) gives \( \mathbb{P}_{2\alpha} \).
Idea of Proof

Truncations

- On the other hand, the truncations we need are obtained by tilting by $\tilde{N}_{t,T} = \mathbb{E}^{\theta_0} \left[ \mathcal{L}_T | \gamma_t \right]$.

- Compute that
  \[
  \tilde{N}_{t,T} = \hat{\mathcal{L}}_t \Psi_t \mathbb{E}^{\theta_t}[\mathcal{L}_{T-t}].
  \]

- Prove an exponential rate of convergence:
  \[
  \mathbb{E}^{\theta_0}[\mathcal{L}_T] = e^{-2an\beta t} \mathcal{F}_a(\theta)[1 + O(e^{-uT})].
  \]

- Compare with the previous calculation. □
Open Questions

- Understanding $4 < \kappa < 8$ (loop interp does not apply, but Bessel results do)
- Take the number of curves to infinity? Use random matrix theory.
- How to unify loop-measure approach with PDE/pure partition function approach to multiple SLE?

Thank you!