Loewner Dynamics for Real Rational Functions and the Multiple $\text{SLE}(0)$ Process

Tom Alberts
Joint with Nam-Gyu Kang (KIAS) and Nikolai Makarov (Caltech)
Multiple SLE(0)

Fix points \( x = \{x_1, \ldots, x_{2n}\}, x_i \in \mathbb{R} \), and a non-crossing link pattern \( \alpha \)

Multiple SLE(0; \( x; \alpha \)) is a special ensemble of \( n \) smooth curves in \( \mathbb{H} \) that connect the points in \( x \) according to pattern \( \alpha \), introduced by Peltola-Wang
**Multiple SLE\((0)\)**

Fix points \(x = \{x_1, \ldots, x_{2n}\}, x_i \in \mathbb{R}\), and a non-crossing link pattern \(\alpha\).

Multiple SLE\((0; x; \alpha)\) is a special ensemble of \(n\) smooth curves in \(\mathbb{H}\) that connect the points in \(x\) according to pattern \(\alpha\), introduced by Peltola-Wang.

\[ C_n = \frac{1}{n+1} \binom{2n}{n} \]
Multiple SLE\((\kappa)\) Motivation
Multiple $\text{SLE}(\kappa)$ as $\kappa \to 0$

Scaling limit of interfaces with alternating boundary conditions
Multiple $\text{SLE}(\kappa)$ as $\kappa \to 0$

Scaling limit of interfaces with *alternating boundary conditions*
Multiple SLE(κ) as κ → 0

• **Kozdron-Lawler:** Reweighting of independent SLEs by a Radon-Nikodym derivative of the form

\[
\exp \left\{ \frac{c(\kappa)}{2} m_{\mathbb{H},\alpha}(\gamma_1, \ldots, \gamma_n) \right\} 1 \{ \gamma_i \cap \gamma_j = \emptyset \text{ for all } i \neq j \}
\]

where \( m_{\mathbb{H},\alpha} \) involves **Brownian loop measure**

• \( c(\kappa) \sim -\frac{12}{\kappa} \) as \( \kappa \to 0 \), so well set up for large deviations

• Extract **multichordal Loewner energy** \( \mathcal{H}(x;\alpha) \) as rate function

• See Eveliina’s talk of last Friday
Properties of Multiple SLE\(^{(0)}\)

Peltola-Wang prove the following properties of multiple SLE\( (0; x; \alpha) \):

- it is the deterministic limit of multiple SLE\( (\kappa; x; \alpha) \) as \( \kappa \to 0 \),
- is the minimizer of the multichordal Loewner energy,
- has the geodesic multichord property,
- can be generated by a Loewner flow involving \( \mathcal{U} = \mathcal{U}(x; \alpha) \),
- is the real locus of a real rational function.
Connection to Real Rational Functions

Peltola-Wang establish the connection to real rational functions by

- showing the functional $\mathcal{H}(x;\alpha)$ has a unique minimizer
- the structure of the functional implies that the minimizing curves have the geodesic multichord property
- Schwarz reflection + the geodesic multichord property implies the curves are the real locus of a real rational function
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Properties of Multiple SLE$^0$

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This talk explains how to generate real locus of a real rational function via Loewner flow, without external inputs

SLE, Brownian loop measure, **conformal field theory** are lurking in many of the ideas, but not in the presentation or the proofs.
Properties of Multiple SLE(0)

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Properties of Multiple SLE(0)

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• is the minimizer of the multichordal Loewner energy
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Alberts-Kang-Makarov:

• new description of Loewner flow for real locus of a real rational function
• alternative formula for the functions \mathcal{U} = \mathcal{U}(x; \alpha)
• direct proof that real loci satisfy geodesic multichord property
• Loewner flow is an instance of Calogero-Moser integrable system
Real Rational Functions and their Real Loci
Real Rational Functions

- Focus on real rational functions of degree \( n + 1 \)
- Ratios of the form
  \[
  \frac{P(z)}{Q(z)}
  \]
  where \( P, Q : \hat{\mathbb{C}} \to \hat{\mathbb{C}} \) are polynomials with real coefficients and
  \[
  \max\{\deg P, \deg Q\} = n + 1
  \]
- Real Locus:
  \[
  \Gamma(R) := \left\{ z \in \hat{\mathbb{C}} : R(z) \in \hat{\mathbb{R}} \right\}
  \]
Real Rational Functions

Basic properties of $\Gamma(R)$:

- $\hat{\mathbb{R}} \subset \Gamma(R)$
- $\Gamma(R)$ is symmetric under conjugation
- $\Gamma(R)$ is path connected
- $\Gamma(R)$ is the union of disjoint arcs
- Number of arcs is related to the degree
- Arcs meet at branch points/critical points

Critical points of $R$: $\{z \in \hat{\mathbb{C}} : R'(z) = 0\}$
Real Rational Functions

- In this talk we specify the critical points and assume the existence of an $R$ with those critical points.
- Restrict to the case of $2n$ real critical points $\mathbf{x} = \{x_1, \ldots, x_{2n}\}$, so
  $$R'(z) = 0 \iff z \in \{x_1, \ldots, x_{2n}\}$$
- **Notation:** $\text{CRR}_{n+1}(\mathbf{x})$ is the set of degree $n + 1$ real rational functions with $2n$ critical points at $\mathbf{x} = \{x_1, \ldots, x_{2n}\}$.
- Degree $n + 1$ plus $2n$ critical points means each critical point of index 2
  $$R(z) = R(x_i) + C(z - x_i)^2 + \ldots$$
  so real locus is locally a $\pm$ shape near each $x_i$.
Real Rational Functions

Structure of Real Locus:  If there exists $R \in \text{CRR}_{n+1}(x)$ the real locus is
- the real line $\mathbb{R}$,
- $n$ non-crossing curves in $\mathbb{H}$ connecting points in $x$
- the complex conjugates of those curves
Real Rational Functions

**Structure of Real Locus:** If there exists $R \in \text{CRR}_{n+1}(x)$ the real locus is

- the real line $\mathbb{R}$,
- $n$ non-crossing curves in $\mathbb{H}$ connecting points in $x$
- the complex conjugates of those curves
Enumerative Algebraic Geometry

• Given $2n$ distinct real points $x$ does there exist any $R \in \text{CRR}_{n+1}(x)$?
• If so how many?

$\phi \circ R = aP + bQ + cP + dQ \in \text{CRR}_{n+1}(x)$ where $\phi(z) = (az + b)/(cz + d)$ is a Möbius transform of $H$ to itself.

Can easily compute that $\Gamma(\phi \circ R) = \Gamma(R)$, so enumeration is done up to equivalence under post-composition by $\text{PSL}(2, \mathbb{R})$.
Enumerative Algebraic Geometry

- Given $2n$ distinct real points $x$ does there exist any $R \in \mathbb{C} \mathbb{R}^{n+1}(x)$?
- If so how many?
- Note once you have one $R \in \mathbb{C} \mathbb{R}^{n+1}(x)$ you have infinitely many since

$$\phi \circ R = \frac{aP + bQ}{cP + dQ} \in \mathbb{C} \mathbb{R}^{n+1}(x)$$

where $\phi(z) = (az + b)/(cz + d)$ is a Möbius transform of $\mathbb{H}$ to itself
- Can easily compute that $\Gamma(\phi \circ R) = \Gamma(R)$, so enumeration is done up to equivalence under post-composition by $\text{PSL}(2, \mathbb{R})$
Enumerative Algebraic Geometry

- Given $2n$ distinct real points $x$ does there exist any $R \in \text{CRR}_{n+1}(x)$?
- If so how many?

Goldberg (one.pnum/nine.pnum/nine.pnum/one.pnum):
For each fixed $x$ the number of equivalence classes is at most $C_n = 1 + (2^n n)$

Eremenko-Gabrielov (two.pnum/zero.pnum/zero.pnum/two.pnum)/Mukhin-Tarasov-Varchenko (two.pnum/zero.pnum/zero.pnum/nine.pnum)/Peltola-Wang (two.pnum/zero.pnum/two.pnum/one.pnum):
For each fixed $x$ there are exactly $C_n$ distinct equivalence classes.
Enumerative Algebraic Geometry

- Given $2n$ distinct real points $x$ does there exist any $R \in CRR_{n+1}(x)$?
- If so how many?
- **Goldberg (1991):** For each fixed $x$ the number of equivalence classes is at most
  \[
  C_n = \frac{1}{n+1} \binom{2n}{n}
  \]
- **Eremenko-Gabrielov (2002) / Mukhin-Tarasov-Varchenko (2009) / Peltola-Wang (2021):** For each fixed $x$ there are exactly $C_n$ distinct equivalence classes.
Loewner Dynamics for Real Rational Functions
Loewner Dynamics for Real Rational Functions

- AKM shows how to grow $\Gamma(R)$ via Loewner flow
- Heavy use of the **poles** of the rational function
- Our results hold *conditionally* on existence of $R \in CRR_{n+1}(x)$
- Our results do *not* use SLE or Brownian loop measure, but the dynamics of our Loewner flow are best described using SLE terminology
SLE(0, x, ρ) Loewner Flows

Definition (SLE(0, x, ρ) Loewner Flows)
Let \( x \in \mathbb{R} \) and \( ρ \) be a finite atomic measure on \( \hat{\mathbb{C}} \) that is symmetric under conjugation. Assume \( ρ(x) = 0 \). SLE(0, x, ρ) is the Loewner flow

\[
\partial_t g_t(z) = \frac{2}{g_t(z) - x_t}, \quad g_0(z) = z,
\]

where the driving function evolves as

\[
\dot{x}_t = \int_{\hat{\mathbb{C}}} \frac{d\rho(w)}{x_t - g_t(w)}, \quad x_0 = x.
\]

Note it matches the standard definition of SLE(κ, x, ρ)

\[
dx_t = \sqrt{\kappa} dB_t + \int_{\hat{\mathbb{C}}} \frac{d\rho(w)}{x_t - g_t(w)} dt, \quad x_0 = x
\]
**SLE**(0, x, ρ) Loewner Flows

Can superimpose flows coming from different x and ρ

**Definition (Superposition of Flows)**

Let $\nu_i : [0, \infty) \rightarrow [0, \infty)$, $i = 1, \ldots, N$ be measurable. The $\nu$-superposition of the SLE(0, $x_j$, $\rho_j$) processes is the superposition of the corresponding flows:

$$
\partial_t g_t(z) = \sum_{j=1}^{N} \frac{2\nu_j(t)}{g_t(z) - x_j(t)} , \quad g_0(z) = z,
$$

where the driving functions $x_1(t), \ldots, x_N(t)$ evolve as

$$
\dot{x}_j = \nu_j(t) \int_{\mathbb{C}} \frac{d\rho_j(w)}{x_j - g_t(w)} + \sum_{k \neq j} \frac{2\nu_k(t)}{x_j - x_k}, \quad x_j(0) = x_j.
$$
**SLE(0, x, ρ) Loewner Flows**

Examples of superpositions

- Grow only the curve anchored at $x_j$:
  \[ \nu_j \equiv 1, \ \nu_k \equiv 0 \text{ for } k \neq j \]

- Grow all curves simultaneously:
  \[ \nu_j \equiv 1 \text{ for all } j \]

- “Adaptive” growth: vary $\nu_j$ with $t$ depending on previous growth
Theorem (Alberts-Kang-Makarov 2020)

Let $x = \{x_1, \ldots, x_{2n}\}$ be distinct real points. Assume $\zeta = \{\zeta_1, \ldots, \zeta_{n+1}\} \subset \hat{\mathbb{C}}$ is closed under conjugation and solves the stationary relation

$$\sum_{j=1}^{2n} \frac{1}{\zeta_k - x_j} = \sum_{l \neq k} \frac{2}{\zeta_k - \zeta_l}, \quad \zeta_k \in \zeta.$$

Then there exists an $R \in \text{CRR}_{n+1}(x)$ with pole set $\zeta$. Moreover, for

$$\rho_j = \sum_{k \neq j} 2 \delta_{x_k} - \sum_{l=1}^{n+1} 4 \delta_{\zeta_l}, \quad j = 1, \ldots, 2n,$$

the curves generated by any $\nu$-superposition of the $\text{SLE}(0, x_j, \rho_j)$ Loewner flows are a subset of $\Gamma(R)$. 

Loewner Dynamics for Real Rational Functions
Theorem says the curves generated by any $\nu$-superposition are a subset of $\Gamma(R)$, which is a form of commutation.

Dynamics remain well-defined as long as each SLE$(0, x_j, \rho_j)$ process is well-defined or until two driving functions $x_j(t)$ collide.

Dynamics may be extendable past the collision times, and in our situation there is a natural way of doing this.
Loewner Dynamics for Real Rational Functions

Two main ingredients: stationary relation + associated $\rho_j$
Loewner Dynamics for Real Rational Functions

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- Stationary relation

$$\sum_{j=1}^{2n} \frac{1}{\zeta_k - x_j} = \sum_{l \neq k} \frac{2}{\zeta_k - \zeta_l}, \quad \zeta_k \in \zeta,$$

and $\zeta = \{\zeta_1, \ldots, \zeta_{n+1}\}$ are poles of a $R \in \text{CRR}_{n+1}(\mathbf{x})$
Loewner Dynamics for Real Rational Functions

Two main ingredients: stationary relation + associated $\rho_j$

- Stationary relation

$$\sum_{j=1}^{2n} \frac{1}{\zeta_k - x_j} = \sum_{l \neq k} \frac{2}{\zeta_k - \zeta_l}, \quad \zeta_k \in \zeta,$$

and $\zeta = \{\zeta_1, \ldots, \zeta_{n+1}\}$ are poles of a $R \in \mathbb{C}RR_{n+1}(x)$

- Any $R = P/Q \in \mathbb{C}RR_{n+1}(x)$ must have $n + 1$ poles
- Complex poles appear in conjugate pairs
- All poles are on the real locus $\Gamma(R)$
- May be a pole at infinity
- Poles are not preserved under post-composition of $R$
Real Rational Functions

Appearance of stationary relation comes from a basic complex analysis result

**Theorem**

Let \( x = \{x_1, \ldots, x_{2n}\} \) be distinct real points, and \( \zeta = \{\zeta_1, \ldots, \zeta_{n+1}\} \subset \mathbb{C} \) be closed under conjugation and distinct from \( x \). There exists an \( R \in CRR_{n+1}(x) \) with pole set \( \zeta \) iff \( \zeta \) solves the stationary relation.

Proof is based on partial fraction expansion of \( R' \). Stationary relation is equivalent to \( R' \) having no residues at the poles \( \zeta_k \).

**Of note:** Generating solutions to the stationary relation is another way of proving existence of \( R \in CRR_{n+1}(x) \)
Loewner Dynamics for Real Rational Functions

• Inserting poles into \( \rho_j = \sum_{k \neq j} 2\delta x_k - \sum_l 4\zeta_l \) gives

\[
\dot{x}_j = U_j(R) := \sum_{k \neq j} \frac{2}{x_j - x_k} + \sum_{\zeta_k \in \zeta} \frac{4}{\zeta_k - x_j}, \quad j = 1, \ldots, 2n.
\]

• Can show that the value of \( U_j \) is invariant under post-compositions of \( R \)
Loewner Dynamics for Real Rational Functions

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• Peltola-Wang has a theorem for Loewner flow but less explicit formula for $U_j$
Loewner Dynamics for Real Rational Functions

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• In Peltola-Wang $U_j(x) = \partial_{x_j} U_\alpha(x)$, where $U_\alpha(x)$ is the minimal value of the **multichordal Loewner potential**

• Minimal value involves the Brownian loop measure of $\Gamma(R)$ and so is difficult to evaluate explicitly
Loewner Dynamics for Real Rational Functions

**Theorem (Alberts-Kang-Makarov 2020)**

Functions $U_j$ satisfy $U_j = \partial_{x_j} \log Z$ where

$$Z(x) = \prod_{1 \leq j < k \leq 2n} (x_j - x_k)^2 \prod_{1 \leq l < m \leq n} (\zeta_l(x) - \zeta_m(x))^8 \prod_{k=1}^{2n} \prod_{l=1}^{n} (x_k - \zeta_l(x))^{-4}$$

Conclusion is that $Z = Ce^U$, but there is no direct proof (as of yet)
Solutions to Null Vector Equations

Using the explicit formula for $U_j$ we are able to show that they solve a system of quadratic equations we call the **null vector equations**

**Theorem (Alberts-Kang-Makarov 2020)**

Let $R \in \mathbb{C}R_{n+1}(x)$ and

$$U_j(x) = U_j(R) := \sum_{k \neq j} \frac{2}{x_j - x_k} + \sum_{\zeta_k \in \zeta} \frac{4}{\zeta_k - x_j}, \quad j = 1, \ldots, 2n.$$ 

The functions $U_j$ solve the system of quadratic equations

$$\frac{1}{2} U_j^2 + \sum_{k \neq j} \frac{2}{x_k - x_j} U_k - \sum_{k \neq j} \frac{6}{(x_k - x_j)^2} = 0, \quad j = 1, \ldots, 2n$$

**Bernard-Bauer-Kytölla (2005): Classical limit of the BPZ equations**
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Using the explicit formula for $U_j$ we are able to show that they solve a system of quadratic equations we call the **null vector equations**

**Theorem (Alberts-Kang-Makarov 2020)**

Let $R \in \mathbb{C} \mathbb{R}_{n+1}(\alpha)$ and

$$U_j(\alpha) = U_j(R) := \sum_{k \neq j} \frac{2}{x_j - x_k} + \sum_{\zeta \in \zeta} \frac{4}{\zeta_k - x_j}, \quad j = 1, \ldots, 2n.$$  

The functions $U_j$ solve the system of quadratic equations

$$\frac{1}{2} U_j^2 + \sum_{k \neq j} \frac{2}{x_k - x_j} U_k - \sum_{k \neq j} \frac{6}{(x_k - x_j)^2} = 0, \quad j = 1, \ldots, 2n$$

**A-Kang-Makarov (2022+):** Commutation + conformal invariance $\implies$ NVE
Solutions to Null Vector Equations

Proof (Null Vector Equations):

\[ R' \text{ has a rational primitive implies the stationary relation} \]
\[ \sum_{j=1}^{n} 1 - x_j = \sum_{l \neq k} 2 \zeta_k - \zeta_l, \zeta_k \in \zeta \]

Based on partial fraction expansion of \( R' \):

\[ R = \frac{P}{Q} \Rightarrow R'(z) = \frac{P'}{Q} - \frac{PQ'}{Q^2}(z) = \prod_{i=1}^{2n} (z - x_i) \prod_{k} (z - \zeta_k)^2 \]
Solutions to Null Vector Equations

Proof (Null Vector Equations):

- $R'$ has a rational primitive implies the stationary relation

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\sum_{j=1}^{2n} \frac{1}{\zeta_k - x_j} = \sum_{l \neq k} \frac{2}{\zeta_k - \zeta_l}, \quad \zeta_k \in \zeta
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Solutions to Null Vector Equations

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• Based on partial fraction expansion of $R'$:

$$R = \frac{P}{Q} \implies R'(z) = \frac{P'Q - PQ'}{Q^2} (z) = \frac{\prod_{i=1}^{2n} (z - x_i)}{\prod_k (z - \zeta_k)^2}$$
Solutions to Null Vector Equations

Proof (Null Vector Equations):

- $R'$ has a rational primitive implies the **stationary relation**

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\sum_{j=1}^{2n} \frac{1}{\zeta_k - x_j} = \sum_{l \neq k} \frac{2}{\zeta_k - \zeta_l}, \quad \zeta_k \in \zeta
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$$

- algebra
Loewner Flow for Real Rational Functions

Summary:

• Expressing Loewner vector fields in terms of poles is beneficial
• Simple formulas for Loewner flow vector fields and solutions to null vector equations
• Our statements and proofs are not probabilistic, but idea of looking at the poles is motivated by Gaussian free field based Conformal Field Theory
Multiple SLE$(0)$ and Calogero-Moser Dynamics
Classical Calogero-Moser Dynamics

One-dimensional many-body problem that is **integrable** and **solvable**

\[ \ddot{x}_j = \sum_{k \neq j} \frac{2}{(x_j - x_k)^3} \]

Hamiltonian system with Hamiltonian given by

\[ \mathcal{H}(x, p) = \frac{1}{2} \sum_j p_j^2 + \sum_{j < k} \frac{1}{(x_j - x_k)^2} \]

Leads to the standard equations of motion

\[ \dot{x}_j = \frac{\partial \mathcal{H}}{\partial p_j} = p_j, \quad \dot{p}_j = \ddot{x}_j = \frac{2}{(x_j - x_k)^3} \]
Classical Calogero-Moser Dynamics

Velocities

![Graph showing velocities over time]

- Axes: Time (x-axis) vs. Velocity (y-axis)
- Several curves indicating the behavior of velocities over time
Calogero-Moser Dynamics for SLE(0)

Theorem (Alberts-Kang-Makarov 2020)
Let \( x = \{x_1, \ldots, x_{2n}\} \) be distinct real points and \( \zeta = \{\zeta_1, \ldots, \zeta_{n+1}\} \subset \hat{\mathbb{C}} \) be closed under conjugation and solve the stationary relation. Under the \(1/4\)-superposition of the SLE\((0, x_j, \rho_j)\) processes

\[
\ddot{x}_j = -\sum_{k \neq j} \frac{2}{(x_j - x_k)^3}.
\]
Calogero-Moser Dynamics for SLE(0)

1/4-superposition of SLE(0, x_j, p_j) gives the coupled system

\[
\dot{x}_j = \sum_{k \neq j} \frac{1}{x_j - x_k} - \sum_k \frac{1}{x_j - \zeta_k},
\]

\[
\dot{\zeta}_k = -\sum_{l \neq k} \frac{1}{\zeta_k - \zeta_l} + \sum_j \frac{1}{\zeta_k - x_j}.
\]

Differentiating and stationary relation leads to Calogero-Moser
Calogero-Moser Dynamics for SLE(0)

$1/4$-superposition of SLE$(0, x_j, \rho_j)$ gives the coupled system

\[
\dot{x}_j = \sum_{k \neq j} \frac{1}{x_j - x_k} - \sum_k \frac{1}{x_j - \zeta_k},
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Calogero-Moser Dynamics for SLE(0)

1/4-superposition of SLE(0, x_j, ρ_j) gives the coupled system

\[ \dot{x}_j = \sum_{k \neq j} \frac{1}{x_j - x_k} - \sum_k \frac{1}{x_j - \zeta_k}, \]

\[ \dot{\zeta}_k = -\sum_{l \neq k} \frac{1}{\zeta_k - \zeta_l} + \sum_j \frac{1}{\zeta_k - x_j}. \]

Gives two ways of describing the Loewner evolution that generates \( \Gamma(R) \):

- as two coupled first order systems of ODEs, with the underlying vector field determined by the poles and critical points of \( R \), and
- via an autonomous second order Calogero-Moser system for the critical points that has no reference to the poles, but must be started with very particular initial momenta.
Calogero-Moser Dynamics for SLE(0)

Generate curves via

$$\partial_t g_t(z) = \sum_{j=1}^{2n} \frac{1/2}{g_t(z) - x_j(t)}, \quad \ddot{x}_j = -\sum_{k \neq j} \frac{2}{(x_j - x_k)^3}$$

Theorem (Alberts-Kang-Makarov /two.pnum/zero.pnum/two.pnum/two.pnum+)

Necessary condition for generated curves to be in the real locus of a real ratio-

satisfy, for $j = 1, \ldots, 2n$,

$$\dot{x}_j^2 - \sum_{k \neq j} \dot{x}_j + x_k - \sum_{k \neq j} \frac{1}{2} \left( x_j - x_k \right)^2 + \frac{1}{2} \sum_{k \neq j} \sum_{l \neq k} \left( x_j - x_k \right) \left( x_j - x_l \right) = 0$$

Show this condition is preserved under Hamiltonian /uniFB02ow using the

Lax pair /three.pnum/two.pnum//three.pnum/two.pnum
Calogero-Moser Dynamics for SLE(0)

Generate curves via

\[ \partial_t g_t(z) = \sum_{j=1}^{2n} \frac{1/2}{g_t(z) - x_j(t)}, \quad \ddot{x}_j = -\sum_{k \neq j} \frac{2}{(x_j - x_k)^3} \]

**Theorem (Alberts-Kang-Makarov 2022+)**

Necessary condition for generated curves to be in the real locus of a real rational function is that \( x(0), \dot{x}(0) \) satisfy, for \( j = 1, \ldots, 2n \),

\[
\dot{x}_j^2 - \sum_{k \neq j} \frac{\dot{x}_j + \dot{x}_k}{x_j - x_k} - \sum_{k \neq j} \frac{1}{(x_j - x_k)^2} + \frac{1}{2} \sum_{k \neq j} \sum_{l \neq k} \frac{1}{(x_j - x_k)(x_j - x_l)} = 0,
\]
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Show this condition is preserved under Hamiltonian flow using the Lax pair.
Geodesic Multichord Property
Theorem (Alberts-Kang-Makarov 2022+)
If \( R \in \text{CRR}_{n+1}(x) \) then \( \Gamma(R) \) satisfies the geodesic multichord property.

“Converse” to Peltola-Wang argument that geodesic multichords are the real loci of real rational functions
Geodesic Multichord Property

Proof is inductive and based on a preservation property of Loewner flows

**Theorem (Alberts-Kang-Makarov 2022+)**

Let \( R \in \text{CRR}_{n+1}(x) \) and \( \zeta_1, \ldots, \zeta_{n+1} \) be the poles of \( R \). Then

\[
R \circ g_t^{-1} \in \text{CRR}_{n+1}(x(t))
\]

for any \( \nu \)-superposition of the SLE\((0, x_j, \rho_j) \) flows

This is a consequence of our integral of motion result
Geodesic Multichord: Pictorial Proof

14 points chosen as critical points
Geodesic Multichord: Pictorial Proof

Real locus of any $R \in \text{CRR}_8(x)$ consists of 7 curves connecting the 14 points in some non-crossing way.
Want to show the pink highlighted curve is a hyperbolic geodesic in the shaded region. Use a Möbius inversion to expose pink curve to $\infty$. 

Geodesic Multichord: Pictorial Proof
Image of curves under Möbius inversion $\phi$. Is also $\Gamma(R \circ \phi^{-1})$, noting that $R \circ \phi^{-1} \in \text{CRR}_8(\phi(x))$. 

Geodesic Multichord: Pictorial Proof
Use superposition of SLE(0, x_j, ρ_j) flows to grow curves that border the shaded region. Previous theorem shows that \( R \circ \phi^{-1} \circ g_t^{-1} \) is also a real rational function. Limiting argument shows \( R \circ \phi^{-1} \circ g_{\tau}^{-1} \) is also real rational function with only two critical points, where \( \tau \) is the time at which all green curves simultaneously complete.
Geodesic Multichord: Pictorial Proof

\[ \Gamma(R \circ \phi^{-1} \circ g_\tau^{-1}) \], which is also the image of the original pink curve under \( g_\tau \circ \phi \). Since \( R \circ \phi^{-1} \circ g_\tau^{-1} \) is real rational with only two critical points can directly compute that it is the hyperbolic geodesic in \( \mathbb{H} \). Conformal invariance of hyperbolic geodesics completes the proof.
Conformal Field Theory Motivation
From CFT to Real Rational Functions

Two main ideas

• $\kappa \to 0^+$ limits of the method of screening for solutions to BPZ equations
• Gaussian free field as a martingale observable for the multiple SLE($\kappa; x; \alpha$) process under Loewner evolution, and its $\kappa \to 0^+$ limit
Multiple SLE($\kappa$) Driving Functions

A single arm of a multiple SLE($\kappa$) ensemble has driving function

$$dx_j(t) = \sqrt{\kappa} dB_t + \kappa(\partial_{x_j} \log Z)(x(t)), \quad x_k(t) = g_t(x_k)$$

Dubédat: Commutation $\Rightarrow Z = Z(x; \kappa)$ solves the BPZ equations: a system of $2n$ linear partial differential equations

Two standard ways of constructing solutions $Z = Z(x)$:

- reweighting independent SLEs via Brownian loop measure terms (Lawler, Lawler/Kozdron)
- contour integration/the method of screening/Coulomb gas integrals (Dubédat, Flores/Kleban, Kytölla/Peltola)

Large deviations come from limits of $\kappa \partial_{x_j} \log Z$ as $\kappa \to 0$
Solutions via Method of Screening

Introduce $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ and define

$$Z^p(x, \zeta) = \prod_{i \neq j} (x_i - x_j)^{2/\kappa} \prod_{i \neq j} (\zeta_i - \zeta_j)^{8/\kappa} \prod_{i,j} (x_i - \zeta_j)^{-4/\kappa}$$
Solutions via Method of Screening

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For appropriate contours $C_1, \ldots, C_n$

$$x \mapsto \oint_{C_1} \cdots \oint_{C_n} Z^p(x, \zeta) \, d\zeta_1 \cdots d\zeta_n =: Z(x; \kappa)$$

solves the BPZ equations.
Solutions via Method of Screening

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For appropriate contours $C_1, \ldots, C_n$

$$x \mapsto \oint_{C_1} \cdots \oint_{C_n} Z^p(x, \zeta) \, d\zeta_1 \cdots d\zeta_n =: Z(x; \kappa)$$

solves the BPZ equations. Ideal for **steepest descent**. As $\kappa \to 0$ integrals concentrate on poles of the rational function

$$\kappa \log Z(x; \kappa) \xrightarrow{\kappa \to 0} \sum_{i \neq j} \log(x_i - x_j)^2 + \sum_{i \neq j} \log((\zeta_i(x) - \zeta_j(x))^8 + \sum_{i,j} \log(x_i - \zeta_j(x))^{-4}$$

and $U_j = \partial_{x_j}$ (RHS)
Method of Screening and Integrals of Motion

Given $x, \zeta$ and $z \in \mathbb{H}$

$\Phi(z; x, \zeta; \kappa) = \text{harmonic extension of boundary conditions determined by } x, \zeta, \kappa$

Then it can be shown that

$$\oint_{C_1} \ldots \oint_{C_n} \Phi(z; x, \zeta; \kappa) \, d\zeta \bigg| g_t^{-1}$$

is a martingale for the multiple SLE($\kappa; x$) process

Apply stationary phase to get an integral of motion for multiple SLE($0; x; \alpha$)
Method of Screening and Integrals of Motion

**Theorem (Alberts-Kang-Makarov 2020)**

For any $\nu$-superposition of SLE$(0, x_j, \rho_j)$ processes the quantities

$$g'_t(z) \frac{\prod_{j=1}^{2n} (g_t(z) - x_j(t))}{\prod_{k=1}^{n+1} (g_t(z) - g_t(\zeta_k(x)))^2}$$

are integrals of motion, for each $z \in \mathbb{H}$.

This does not require $\zeta$ to satisfy the stationary relation.

When stationary relation is satisfied, integral of motion is key in the proof that generated hull is in $\Gamma(R)$.

Integral of motion also leads to **geodesic multichord property** for $\Gamma(R)$.
Open Questions
Critical Points of Higher Multiplicities

Throughout we’ve assumed \( R = P/Q \) is real rational with critical points \( x = (x_1, \ldots, x_{2n}) \), and critical points are order 1, i.e.

\[
R'(z) \sim C(z - x_i), \quad z \to x_i
\]

What about higher multiplicities?
Critical Points of Higher Multiplicities

Throughout we've assumed $R = P/Q$ is real rational with critical points $x = (x_1, \ldots, x_{2n})$, and critical points are order $1$, i.e.

$$R'(z) \sim C(z - x_i), \quad z \to x_i$$

What about higher multiplicities?
Natural Time Parameterizations

\[ \partial_t g_t(z) = \sum_{j=1}^{2n} \frac{2\nu_j(t)}{g_t(z) - x_j(t)}, \quad g_0(z) = z, \quad x_j(0) = x_j \]

where \( x(t) = (x_1(t), \ldots, x_{2n}(t)) \) evolve according to

\[ \dot{x}_j(t) = U_j(R \circ g_t^{-1})\nu_j(t) + \sum_{k \neq j} \frac{2\nu_k(t)}{x_j(t) - x_k(t)}, \quad j = 1, \ldots, 2n. \]
Fix $x = (x_1, \ldots, x_{2n})$. Up to permutation of coordinates, how many solutions $\zeta = (\zeta_1, \ldots, \zeta_n) \in \mathbb{C}^n$ are there to the stationary relation

$$\sum_{j=1}^{2n} \frac{1}{\zeta_k - x_j} = \sum_{l \neq k}^{2} \frac{2}{\zeta_k - \zeta_l}, \quad k = 1, \ldots, n?$$

Goldberg + Eremenko-Gabrielov/Peltola-Wang implies there should be exactly $C_n$, i.e. solutions are enumerated by link patterns.

Is there a way to generate all solutions from one particular solution?
Fix $x = (x_1, \ldots, x_{2n})$. Can one directly enumerate the number of (real) solutions $(U_1, \ldots, U_{2n})$ to the null vector equations

$$\frac{1}{2} U_j^2 + \sum_{k \neq j} \frac{2}{x_k - x_j} U_k - \sum_{k \neq j} \frac{6}{(x_k - x_j)^2} = 0, \quad j = 1, \ldots, 2n$$

and conformal Ward identities

$$\sum_{j=1}^{2n} U_j = 0, \quad \sum_{j=1}^{2n} x_j U_j = -6n, \quad \sum_{j=1}^{2n} x_j^2 U_j = -6 \sum_{j=1}^{2n} x_j.$$
Actual Pictures
Multiple $\text{SLE}(0)$
Multiple SLE\(^{(0)}\)