Geodesics in the Brownian map: Strong confluence and geometric structure

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Plan

1. Background and motivation
2. Strong confluence of geodesics
3. Geometric structure of geodesics
4. Approximation by geodesics between typical points
5. Ideas of the proofs
6. Open questions
Background and motivation
The Brownian map

General idea

The Brownian map is the “canonical” model for a metric space chosen “uniformly at random” among metric spaces which have the topology of the two-dimensional sphere $S^2$. 

Denoted by $(S, d, \nu)$.

Homeomorphic to the sphere $S^2$ [Le Gall and Paulin '08] (also see a later proof [Miermont '08])

Hausdorff dimension equal to 4 [Le Gall '07]

Equivalent as a metric measure space to $\sqrt{8/3}$-LQG (Liouville quantum gravity) [Miller and Sheffield '16 and '20], which serves to canonically embed the Brownian map into $S^2$. 

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- Gromov-Hausdorff scaling limit of a large class of planar maps chosen uniformly at random.
  - Triangulations and $2p$-angulations with $n$ faces [Le Gall ’13]
  - Quadrangulations with $n$ faces [Miermont ’13]
  - Bipartite planar maps, random simple triangulations and quadrangulations, ...
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Approximation by quadrangulation

Image by Jérémie Bettinelli
Classification of all geodesics to the root

[Le Gall '10] Let $\rho \in S$ be a distinguished point called the root. The following holds a.s.

- For $\nu$-a.e. point $z \in S$, there is a unique geodesic between $z$ and $\rho$. 

The law of $(S, d, \nu, \rho)$ is invariant if we resample $\rho$ independently according to $\nu$. 

This plays a major role in the works that identify the Brownian map as the scaling limit of uniform random maps [Le Gall] and [Miermont], as well as in the proof of the equivalence of $\sqrt{8}/3$-LQG with the Brownian map [Miller and Sheffield].
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- Every point in $S$ is connected to $\rho$ by at most 3 geodesics. The set of points connected by 2 or 3 geodesics to $\rho$ is dense in $S$ and has $\nu$-mesure zero.
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Geodesics between exceptional points?

**Figure** – A (3, 2)-normal network

The set of pairs of points connected by a \((j, k)\)-normal network is non-empty if and only if \(j, k \in \{1, 2, 3\}\).

The set of pairs of points connected by a \((3, 3)\)-normal network is dense and countable.

However, there exist other exceptional points between which the collection of geodesics has a topology which is not that of a normal network.

AKM also proves a strong version of the confluence of geodesics. This version is also associated with typical points and does not apply to all geodesics.
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\[
\begin{array}{c}
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\end{array}
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- Is there any point from which infinitely many disjoint (except at the starting point) geodesics emanate?

![Figure - A geodesic star](image-url)
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![Geodesic star](image)

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Our goal is to answer these questions and to provide a global description of the behavior of all geodesics at the same time.
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- We show that a different form of the confluence of geodesics phenomenon which holds simultaneously for all geodesics in the Brownian map.

**Definition (Hausdorff distance)**

Let $X$ be a metric space. For all $A \subseteq X$ and $\varepsilon > 0$, let $A(\varepsilon) = \bigcup_{x \in A} B(x, \varepsilon)$ be the $\varepsilon$-neighborhood of $A$. The Hausdorff distance between two closed sets $A, B \subseteq X$ is defined to be

$$d_H(A, B) = \inf \{\varepsilon > 0 : A \subseteq B(\varepsilon), B \subseteq A(\varepsilon)\}.$$
Theorem 1 (Miller, Q. ’20)

The following holds for $\mu_{BM}$ a.e. instance of Brownian map $(S, d, \nu)$. For each $u > 0$, there exists $\varepsilon_0 > 0$ such that for all $\varepsilon \in (0, \varepsilon_0)$, the following holds. Let $\delta = \varepsilon^{1-u}$. Suppose that $\eta_i : [0, T_i] \to S$ for $i = 1, 2$ are two geodesics with $T_i = d(\eta_i(0), \eta_i(T_i)) \geq 2\delta$ and

$$d_H(\eta_1([0, T_1]), \eta_2([0, T_2])) \leq \varepsilon,$$

then

$$\eta_i([\delta, T_i - \delta]) \subseteq \eta_{3-i} \text{ for } i = 1, 2.$$
**Strong confluence of geodesics**

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Strong confluence of geodesics (more precise version)

**Definition (interior-internal metric)**

Let \((X, d)\) be a metric space and \(S \subseteq X\). Let \(d_S\) be the interior-internal metric on \(S\), whereby \(d_S(u, v)\) is given by the infimum of the \(d\)-length of paths which are contained in the interior of \(S\), except possibly their endpoints.

**Definition (One-sided Hausdorff distance)**

Let \(\eta_1, \eta_2\) be two geodesics of \((S, d, \nu)\). Then \(S \setminus \eta_1\) is a simply connected set whose boundary is the union of the left and right sides of \(\eta_1\), which we denote by \(\eta_1^L\) and \(\eta_1^R\). Let \(\ell_L\) (resp. \(\ell_R\)) be the Hausdorff distance between \(\eta_1^L\) (resp. \(\eta_1^R\)) and \(\eta_2 \setminus \eta_1\) with respect to the interior-internal metric \(d_{S \setminus \eta_1}\). We define the one-sided Hausdorff distance from \(\eta_1\) to \(\eta_2\) by

\[
d^1_H(\eta_1, \eta_2) = \min(\ell_L, \ell_R).
\]

We always have

\[
d_H(\eta_1, \eta_2) \leq d^1_H(\eta_1, \eta_2).
\]
Strong confluence of geodesics (more precise version)

Theorem 2 (Miller, Q. '20)

There exists $c > 0$ such that the following holds for $\mu_{BM}$ a.e. instance of Brownian map $(S, d, \nu)$. There exists $\epsilon_0 > 0$ such that for all $\epsilon \in (0, \epsilon_0)$, the following holds. Let $\delta = c \epsilon \log \epsilon - 1$. Suppose that $\eta_i : [0, T_i] \to S$ for $i = 1, 2$ are two geodesics with $T_i = d(\eta_i(0), \eta_i(T_i)) \geq 2 \delta$ and $d_{1H}(\eta_1([0, T_1]), \eta_2([0, T_2])) \leq \epsilon$, then $\eta_1([\delta, T_1 - \delta]) \subseteq \eta_3 - i$ for $i = 1, 2$. 

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Strong confluence of geodesics (more precise version)

- We believe that the order of magnitude $\varepsilon \log \varepsilon^{-1}$ is optimal in Theorem 2.
- Theorem 2 $\implies$ Theorem 1.
  - It is enough to consider the case where $\eta_1$ and $\eta_2$ do not cross each other.
  - There are at most $\varepsilon^{-u}$ bottlenecks along a geodesic.
Geometric structure of geodesics
Intersection behavior of geodesics

**Theorem 3 (Miller, Q. '20)**

The following holds for $\mu_{BM}$ a.e. instance of Brownian map $(S, d, \nu)$. Suppose that $\eta_i : [0, T_i] \to S$ for $i = 1, 2$ are two geodesics, then $\eta_1((0, T_1)) \cap \eta_2((0, T_2))$ is connected.

The following configurations are impossible.

The following configurations are not ruled out.
Geodesic stars

Let $\Psi_k$ be the set of $k$-star points.

\textbf{Figure} – A 5-star point
Geodesic stars

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Theorem 4 (Miller, Q. ’20)

The following holds for $\mu_{BM}$ a.e. instance of Brownian map $(S, d, \nu)$. The set $\Psi_k$ is empty for $k \geq 6$. For $1 \leq k \leq 5$, we have

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- [Miermont ’13] conjectured that there exist $k$-star points for $1 \leq k \leq 4$, and there do not exist $k$-star points for $k \geq 6$. 

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- The matching lower bounds were recently proved by Le Gall.
- It is still an open question whether there exist 5-star points.
Topology of geodesics between a pair of points

- Theorems 3 and 4 together reduce the possible configurations of geodesics between any pair of points to a finite number of cases up to homeomorphism.

Definition (Splitting point)

For $u, v \in S$ distinct, we say that $z$ is a splitting point from $v$ to $u$ of multiplicity at least $k$, if there exist $0 < r < t < d(u,v)$ and geodesics $\eta_1, \ldots, \eta_{k+1}$ from $v$ to $u$ such that $\eta_i(t) = z$ for all $1 \leq i \leq k+1$ and $\eta_i([t-r, t]) \cap \eta_j([t-r, t]) = \emptyset$ for all $1 \leq i < j \leq k+1$. The multiplicity of $z$ is equal to the largest integer $k$ such that the property above holds.
**Topology of geodesics between a pair of points**

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- We will further reduce the number of possible configurations, and then give a **dimension upper bound** for the set of pairs of points connected by each configuration (up to homeomorphism).
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$$\eta_i([t - r, t]) = \eta_j([t - r, t]), \quad \eta_i((t, t + r)) \cap \eta_j((t, t + r)) = \emptyset$$

for all $1 \leq i < j \leq k + 1$. The **multiplicity** of $z$ is equal to the largest integer $k$ such that the property above holds.
Topology of geodesics between a pair of points

**Theorem 5 (Miller, Q. '20)**

The following holds for $\mu_{BM}$ a.e. instance of Brownian map $(S, d, \nu)$. For all $u, v \in S$ distinct, every geodesic from $v$ to $u$ contains at most two splitting points from $v$ to $u$, and the multiplicity of each splitting point is 1. Let $\Phi_{I, J, K}$ be the set of $(u, v)$ such that $u, v \in S$ are distinct and there exists $r > 0$ so that the following holds.

1. There are geodesics $\eta_1, \ldots, \eta_I$ from $u$ to $v$ such that the sets $\eta_i((0, r))$ for $1 \leq i \leq I$ are pairwise disjoint.
2. There are geodesics $\eta_1, \ldots, \eta_J$ from $v$ to $u$ such that the sets $\eta_i((0, r))$ for $1 \leq i \leq J$ are pairwise disjoint.
3. There are $K$ splitting points from $v$ to $u$.

If $11 - (I + 2J + K) \geq 0$, then

$$\dim_H(\Phi_{I, J, K}) \leq 11 - (I + 2J + K).$$

Otherwise $\Phi_{I, J, K} = \emptyset$. 
Figure – Optimal configurations and the associated triplets \((I, J, K)\)
Topology of geodesics between a pair of points

- The asymmetry between $l$ and $J$ in Theorem 5 is due to the asymmetry in the definition of a splitting point.
- In the language of [Angel, Kolesnik and Miermont ’17], if $u$ and $v$ are connected by a $(j, k)$-normal network, then $l = j, J = k$ and $K = j - 1$. Theorem 5 implies that the dimension of such pairs $(u, v)$ is at most

$$11 - (j + 2k + (j - 1)) = 12 - 2(j + k),$$

equal to the dimension computed in [Angel, Kolesnik and Miermont ’17].
Number of geodesics between a pair of points

Let $\Phi_i$ be the set of pairs of distinct points in $S$ that are connected by exactly $i$ geodesics.

### Theorem 6 (Miller, Q. '20)

The following holds for $\mu_{BM}$ a.e. instance of Brownian map $(S, d, \nu)$. The set $\Phi_i$ is empty if $i \geq 10$. For $1 \leq i \leq 9$, we have

$$
\dim_H(\Phi_1) = 8, \quad \dim_H(\Phi_2) = 6, \quad \dim_H(\Phi_3) = 4, \quad \dim_H(\Phi_4) = 4
$$

$$
\dim_H(\Phi_5) = 2, \quad \dim_H(\Phi_6) = 2, \quad \dim_H(\Phi_7) = 0, \quad \dim_H(\Phi_8) = 0, \quad \dim_H(\Phi_9) = 0.
$$

The sets $\Phi_7, \Phi_8, \Phi_9$ are countably infinite. For all $1 \leq i \leq 9$, the set of points $u \in S$ such that there exists $v \in S$ with $(u, v) \in \Phi_i$ is dense in $S$. 
Number of geodesics between a pair of points

The **upper bounds** in Theorem 6 follow from Theorem 5 and the optimal configurations.
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The **lower bounds** in Theorem 6 and the **description of** $\Phi_7, \Phi_8, \Phi_9$ are obtained as follows:

- For $i \in \{2, 3, 4, 6, 9\}$: By [Angel, Kolesnik and Miermont '17], the dimension of the pairs of points connected by a $(j, k)$-normal network is $12 - 2(j + k)$. Since $(j, k)$-normal networks $\subseteq \Phi_{jk}$, this gives the lower bounds of $\dim_H(\Phi_i)$ for $i \in \{2, 3, 4, 6\}$.

It was shown in [Angel, Kolesnik and Miermont '17] that there is a dense and countably infinite set of points connected by a $(3, 3)$-normal network. Theorem 5 shows that there do not exist other configurations leading to 9 geodesics.
The upper bounds in Theorem 6 follow from Theorem 5 and the optimal configurations.

The lower bounds in Theorem 6 and the description of $\Phi_7, \Phi_8, \Phi_9$ are obtained as follows:

- For $i \in \{2, 3, 4, 6, 9\}$: By [Angel, Kolesnik and Miermont ’17], the dimension of the pairs of points connected by a $(j, k)$-normal network is $12 - 2(j + k)$. Since $(j, k)$-normal networks $\subseteq \Phi_{jk}$, this gives the lower bounds of $\dim_H(\Phi_i)$ for $i \in \{2, 3, 4, 6\}$.

- It was shown in [Angel, Kolesnik and Miermont ’17] that there is a dense and countably infinite set of points connected by a $(3, 3)$-normal network. Theorem 5 shows that there do not exist other configurations leading to 9 geodesics.

- For $i \in \{5, 7, 8\}$, the optimal configurations are not normal networks. We will use different techniques to deal with these cases.
Approximation by geodesics between typical points
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**Theorem 7 (Miller, Q. '20)**

The following holds for $\mu_{BM}$ a.e. instance of Brownian map $(S, d, \nu)$. For every geodesic $\eta : [0, T] \rightarrow S$, every $0 < s < t < T$ and $\varepsilon > 0$, there exists $\delta > 0$ such that every geodesic $\xi : [0, S] \rightarrow S$ with $\xi(0) \in B(\eta(s), \delta)$ and $\xi(S) \in B(\eta(t), \delta)$ satisfies

$$\xi([\varepsilon, S - \varepsilon]) \subseteq \eta \text{ et } \eta([s + \varepsilon, t - \varepsilon]) \subseteq \xi.$$
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$$\xi([\varepsilon, S - \varepsilon]) \subseteq \eta \quad \text{et} \quad \eta([s + \varepsilon, t - \varepsilon]) \subseteq \xi.$$

We can choose the points $\xi(0)$ and $\xi(S)$ to be $\nu$-typical, which implies that every geodesic of the Brownian map can be arbitrarily well approximated by a geodesic between typical points.
The geodesic frame $GF(S)$ is the union of all the geodesics in $S$ minus their endpoints.

- Clearly, $\dim_H GF(S) \geq 1$. 

Conjecture: $\dim_H GF(S) = 1$. [Angel, Kolesnik and Miermont '17]

Corollary 8 (Miller, Q. '20) For $\mu_{BM}$ a.e. instance of Brownian map $(S, d, \nu)$, we have $\dim_H GF(S) = 1$. 

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**Corollary 8 (Miller, Q. ’20)**

For $\mu_{\text{BM}}$ a.e. instance of Brownian map $(S, d, \nu)$, we have $\dim H \text{GF}(S) = 1$. 

- Gaussian free field $h$. The color represents the height of $h$.
- The metric of $\sqrt{8/3}$-LQG is given by
  \[ e^{\sqrt{8/3}h(x)}(dx^2 + dy^2). \]
- The length of each path $P$ is given by
  \[ \sum_{x \in P} e^{\sqrt{8/3}h(x)/4}. \]
Ideas of the proofs
Main idea : depth-first vs. breadth-first

The proofs of previous results (e.g. [Le Gall ’10] and [Angel, Kolesnik and Miermont ’17]) primarily make use of the Brownian snake encoding of the Brownian map, see [Chassaing and Schaeffer ’04], [Marckert and Mokkadem ’06] and [Le Gall ’07].

- Analogous to the Cori-Vauquelin-Schaeffer bijection for the quadrangulations.
- The Brownian map is constructed from a labeled continuous random tree (CRT). [Aldous ’91, ’93]
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- The Brownian map is constructed from a labeled continuous random tree (CRT). [Aldous ’91, ’93]

This corresponds to the depth-first exploration of the Brownian map. **This leads to very precise description of the geodesics to the root.**
Main idea: depth-first vs. breadth-first

Our work primarily make use of the **breadth-first** exploration of the Brownian map.

- Analogous to the **peeling by layers** of random planar maps. [Ambjørn, Durhuus, Jonsson and Jonsson ’97], [Watabiki ’95] and [Angel ’03]
- Various aspects in the discret and in the continuum were developed by Bertoin, Budd, Curien, Kortchemski, Le Gall, Miller and Sheffield, and so on.

We will in particular use the setting and results from [Miller and Sheffield ’15] “An axiomatic characterization of the Brownian map”
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**Particularly amenable for establishing independence properties along geodesics.**
The root $x$ and the dual root $y$ are distributed as two independently chosen points in $\mathcal{S}$ according to $\nu$.

This construction gives $\mu_{BM}^{A=1}$. The measure $\mu_{BM}$ is constructed by first choosing the time length of the excursion according to the infinite measure $ct^{-3/2}dt$, and then sampling a Brownian excursion on $[0, t]$.
Breadth-first exploration of the Brownian map

Let \((S, d, \nu, x, y)\) be sampled from \(\mu_{BM}\). Let \(B_y^\bullet(x, r)\) be the metric ball of radius \(r\) centred at \(x\) and filled with respect to \(y\). We can associate a boundary length \(L_r\) to \(\partial B_y^\bullet(x, r)\).

**Fact**

The process \((L_{d(x,y)-r}, 0 \leq r \leq d(x, y))\) is distributed as a continuous state branching process (CSBP) with parameter \(3/2\).
Continuous state branching process (CSBP)

- Introduced in [Jiřina '58], also studied in [Lamperti '67]. Also see the more recent expository texts [Le Gall '99] and [Kyprianou '06].
- It is defined via the Lamperti transform. If \((X_s)\) is an \(\alpha\)-stable Lévy process with only upward jumps and

\[
s(t) = \inf \left\{ r > 0 : \int_0^r \frac{1}{X_u} \, du \geq t \right\},
\]

then \(Y_t := X_{s(t)}\) is an \(\alpha\)-CSBP.
- The transition kernel of \(Y\) satisfies

\[
P_t(x_1 + x_2, \cdot) = P_t(x_1, \cdot) \ast P_t(x_2, \cdot).
\]

- \((Y_{C_{\alpha^{-1} t}})\) is equal in distribution to \((CY_t)\).
- One can also define an excursion measure for \(\alpha\)-stable CSBP by doing the Lamperti transform to an \(\alpha\)-stable Lévy excursion sampled as follows:
  - Pick a lifetime \(t\) from the infinite measure \(t^{-1 - 1/\alpha} dt\)
  - Given \(t\), sample an \(\alpha\)-stable Lévy excursion.

In the Brownian map \((S, d, \nu, x, y)\) sampled from \(\mu_{BM}\), we have \(t = d(x, y)\).
Decomposition into metric bands

- Fix $0 < r_1 < r_2 < \cdots < r_k$. For each $1 \leq j \leq k$, 
  $B_j := B_y^*(x, d(x, y) - r_j) \setminus B_y^*(x, d(x, y) - r_{j+1})$ is a metric space with interior-internal metric $d_{B_j}$ and the measure $\nu_{B_j} := \nu|_{B_j}$.

- On the event $d(x, y) > r_j$, $B_j$ is non-empty, and is either an annulus if $d(x, y) > r_{j+1}$ or a topological disk if $d(x, y) \leq r_{j+1}$.

- $B_j$ is independent of $B_1, \ldots, B_{j-1}$, conditionally on the length of $\partial_{In} B_j$.

- The boundary $\partial_{In} B_j$ is naturally marked by the unique point visited by the unique geodesic between $x$ and $y$. The quantity $r_{j+1} - r_j$ is called the width of $B_j$. 

![Diagram of metric bands](image-url)
Sketch of the proof of strong confluence

Step 1 : A weaker version of the strong confluence.

If two geodesics are sufficiently close with respect to the one-sided Hausdorff distance, then they should intersect each other near their endpoints.
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If two geodesics are sufficiently close with respect to the one-sided Hausdorff distance, then they should intersect each other near their endpoints.

- For two $\nu$-typical points $x, y$, with overwhelming probability, there are many $\mathcal{X}$'s along the geodesic $\eta$ between $x$ and $y$. Every branch of an $\mathcal{X}$ is the **unique** geodesic between its endpoints.
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In each metric band, there is a positive probability that an \( \mathcal{X} \) occurs.
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- If \( \tilde{\eta} \) crosses an \( \mathcal{X} \) centred on \( \eta(t) \), then \( \tilde{\eta} \) also intersects \( \eta(t) \).
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- In each metric band, there is a positive probability that an $\mathcal{X}$ occurs.
- If $\tilde{\eta}$ crosses an $\mathcal{X}$ centred on $\eta(t)$, then $\tilde{\eta}$ also intersects $\eta(t)$.
- If $\tilde{\eta}_1$ and $\tilde{\eta}_2$ are close to each other, then one can find a geodesic $\eta$ between $\tilde{\eta}_1$ and $\tilde{\eta}_2$. 
Open questions
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- Establish the dimension lower bound for the following configurations (the dimension upper bound is given by $11 - (I + 2J + K) \geq 0$).

- Do the six last configurations exist?

- Do there exist 5-star points?

- Geodesics to the boundary of the Brownian disk (work in progress with T. He).

- Recent works [Gwynne '20] and [Gwynne, Pfeffer, Sheffield '20] prove the analogues of [Le Gall '10] and [Angel, Kolesnik, Miermont '17] for the $\gamma$-LQG for $\gamma \in (0, 2)$. The analogue of our results remain open for the LQG. We believe that a proof can be established following the same strategy, using GFF, but things can get even more technical.

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The analogue of our results remain open for the LQG. We believe that a proof can be established following the same strategy, using GFF, but things can get even more technical.
Thank you very much for your attention!