Simply connected wandering domains

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MSRI

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The connected components of the Fatou set, which map into each other, are called **Fatou components**.
Fatou components can be:

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- **periodic**

- **preperiodic**
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- **Periodic**

- **Preperiodic**

- **Wandering domains**
Definition
Let $U$ be a Fatou component of $f$. If $f^n(U) \cap f^m(U) = \emptyset$, for all $m, n \in \mathbb{N}$, with $m \neq n$ then $U$ is a wandering domain.
An example of a wandering domain

Figure 1: The dynamics of the function $f(z) = z + 2\pi \sin z$ (picture by Lasse Rempe).
Three types of wandering domains

Wandering domains are classified into three types with respect to escape to infinity.

1. **Escaping**
   \[ f_n(z) \to \infty \quad \text{for all} \quad z \in U \]

2. **Oscillating**
   \[ f_{n_k}(z) \to \infty \quad \text{and} \quad f_{m_k}(z) \text{ stays bounded for all} \quad z \in U \]

3. **Bounded (orbit)**
   \[ f_n(z) \text{ stays bounded for all} \quad z \in U \]
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Wandering domains are classified into three types with respect to escape to infinity.

- **Escaping**, if $f^n(z) \to \infty$ for all $z \in U$

- **Oscillating**, if there exist $(n_k), (m_k)$ such that $f^{n_k}(z) \to \infty$ and $(f^{m_k}(z))$ stays bounded for all $z \in U$.

- **Bounded (orbit)** if $(f^n(z))$ stays bounded for all $z \in U$. 


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Early results

**Theorem (Sullivan 1984)** Wandering domains do not exist for rational maps.
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**Theorem (Sullivan 1984)** Wandering domains do not exist for rational maps.

Baker (1984) was the first to give an example of a transcendental entire function with a wandering domain. The wandering domain in Baker’s example was multiply connected.

A detailed description of the dynamics in multiply connected wandering domains was given by Bergweiler, Rippon and Stallard in 2011.
Why wandering domains?

In recent years, there is an increased interest in the study of wandering domains.
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- Wandering domains are the least understood of all Fatou components.
- There are several big open questions in Holomorphic Dynamics concerning wandering domains.
This project is in collaboration with
Our project

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Anna Miriam Benini
University of Parma

Nuria Fagella
University of Barcelona

Phil Rippon
The Open University

Gwyneth Stallard
The Open University
We studied the *internal* dynamics in simply connected wandering domains.

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We obtained a nine-way classification of simply connected wandering domains:

- in terms of hyperbolic distances between orbits of points and
- in terms of converging to the boundary.
First classification theorem

**Theorem.** Let $U$ be a simply connected wandering domain and suppose $z, w \in U$ have distinct orbits. Then there are three possibilities.
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1. $U$ is **contracting**: for all such pairs $z, w \in U$, $d_{U_n}(f_n(z), f_n(w))$ decreases to 0.
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1. $U$ is contracting: for all such pairs $z, w \in U$, 
   \[ d_{U_n}(f_n(z), f_n(w)) \] decreases to 0.

2. $U$ is semi-contracting: for all such pairs $z, w \in U$, 
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2. $U$ is **semi-contracting**: for all such pairs $z, w \in U$, $d_{U_n}(f_n(z), f_n(w))$ decreases to $c(z, w) > 0$.

3. $U$ is **eventually isometric**: for all such pairs $z, w \in U$, $d_{U_n}(f_n(z), f_n(w))$ is eventually constant.
Idea of the proof

\[ U_0 \xrightarrow{f} U_1 \xrightarrow{f} \ldots \xrightarrow{f} U_{n-1} \xrightarrow{f} U_n \]

\[ \phi_0 \downarrow \quad \phi_1 \downarrow \quad \phi_{n-1} \downarrow \quad \phi_n \downarrow \]

\[ 0 \xrightarrow{g_1} 0 \xrightarrow{g_2} \ldots \xrightarrow{g_{n-1}} 0 \xrightarrow{g_n} 0 \]

\[ D \xrightarrow{D} D \xrightarrow{D} \ldots \xrightarrow{D} D \xrightarrow{D} D \]

\[ G_n \]

\[ \hat{d} U_n(\tilde{f}_n(z_0), \tilde{f}_n(z_0)) = \hat{d} D(G_n(w), G_n(w)) \rightarrow 0 \Rightarrow \hat{d} D(G_n(w), G_n(w)) \rightarrow 0 \]
Idea of the proof

\[ U_0 \xrightarrow{\phi_0} U_1 \xrightarrow{\phi_1} \cdots \xrightarrow{\phi_{n-1}} U_{n-1} \xrightarrow{\phi_n} U_n \]

\[ \hat{d} U_n(f_n(z), f_n(z_0)) = \hat{d} \mathbb{D}(G_n(w), 0) \]

\[ \hat{d} \mathbb{D}(G_n(w), 0) \rightarrow 0 \Rightarrow \hat{d} \mathbb{D}(G_n(w), G_n(w')) \rightarrow 0 \]
Second classification theorem

Theorem. Let $U$ be a simply connected wandering domain. Then there are three possibilities.

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(b) **bungee** For all $z \in U$, there is a subsequence $f_{n_k}(z)$ which converges to $\partial U_{n_k}$ and a subsequence which stays away; or

(c) **converging** For all $z \in U$, $f_n(z)$ converges to $\partial U_n$. 
The behaviour of two points/one point in the wandering domain determines the type of the wandering domain with respect to the first/second classification.
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For example, if the orbit of one point converges to the boundary of the wandering domain then all internal orbits do.
Possible types of wandering domains

The two classification theorems give rise to 9 possible types of escaping simply connected wandering domains, only 3 of which were known to exist before.
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A wandering domain coming from a lift

Figure 2: The dynamics of the function $f(z) = z + 2\pi + \sin z$ (picture by David Martí-Pete).

The function $g(z) = z \exp\left(\frac{1}{2}\left(\frac{1}{z} - z\right)\right)$ has a superattracting basin, which lifts to a sequence of wandering domains.
All types are realisable!

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More on this construction, as well as a recent exciting construction by Martí-Pete, Rempe and Waterman will be presented during the mini-course ‘Approximation in Transcendental Dynamics’.
More recently, we studied the behaviour of boundary points of simply connected wandering domains in terms of convergence.
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You will hear all about this work in Nuria’s talk at the Introductory workshop.
THANK YOU