The spherical Plateau problem

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The hyperbolic part of 3-manifolds

A consequence of the Geometrization theorem (Thurston, ..., Perelman):

Given $M$ a closed oriented 3-manifold, $M = M_1 \# \ldots \# M_k$ where each $M_j$ is prime. Then after cutting $M_j$ along tori, $M_j = H_j \cup S_j$ where $H_j$ admits a finite volume hyperbolic metric, and $S_j$ is a Seifert manifold (i.e. "partitioned into circles").

The hyperbolic part of $M$ is $M_{\text{hyp}} := \bigcup_{j=1}^k H_j$ (it is unique and has a unique hyperbolic metric $g_{\text{hyp}}$).
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- $M_{hyp}$ is the truly 3-dimensional part of $M$, 

Given any Riemannian metric $(M, g)$, the correctly normalized Ricci flow with surgery converges to $(M_{hyp}, g_{hyp})$ as $t \to \infty$.

We'll see next that $(M_{hyp}, g_{hyp})$ is the unique solution of an infinite codimension Plateau problem.
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Consider a sequence \(\{C_i\} \subset C(h)\) such that

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By Almgren (see also De Lellis-Spadaro), \(C_\infty\) is smooth outside a codimension 2 subset. Call such \(C_\infty\) a Plateau solution for \(h \in H_n(N; \mathbb{Z})\).
The theory of integral currents has been generalized in two directions:

- White considered arbitrary complete normed abelian groups,
- Ambrosio-Kircheim considered any complete ambient metric space (subsequent works of Lang, Wenger, Schmidt, Sormani-Wenger...).
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De Pauw-Hardt developed a very general theory encompassing both directions.
Let $(\mathcal{K}, g)$ be a separable Hilbert manifold of bounded diameter, and let $h \in H_n(\mathcal{K}; \mathbb{Z})$. 
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Because \(\dim(\mathcal{K}) = \infty\), \(C_i\) may not converge in any reasonable way to an integral current inside \(\mathcal{K}\).
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given a sequence of boundaryless integral currents $S_i$ of uniformly bounded masses and diameters, subsequentially there are a Banach space $Z$, an integral current $S_\infty \subset Z$, and isometric embeddings

\[ j_i : S_i \hookrightarrow Z \]

such that inside $Z$,

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Rm: in fact here $S_i$ converges in the intrinsic flat topology (Sormani-Wenger).
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In general, $C_\infty$ is only known to be an integral current. At least for an area-minimizing integral current $T$ in a Hilbert space, Ambrosio-De Lellis-Schmidt showed that $\text{spt}(T)$ is smooth in an open dense subset of $\text{spt}(T)$. An Almgren type theorem seems plausible for such $T$. 
An example of Plateau solution in infinite codimension

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Let \(S_{\partial \tilde{M}} := \) unit sphere in \(L^2(\partial \tilde{M}; \mathbb{Z})\).
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By \(\pi_1(M)\)-equivariance, it gives an embedding

\[
B : (M, \frac{(n - 1)^2}{4n} g_0) \to S_{\partial \tilde{M}} / \pi_1(M).
\]
An example of Plateau solution in infinite codimension

It is an isometric and minimal embedding (checked with a computation). However we have much better:

Besson-Courtois-Gallot’s theorem: the image $B(M)$ is calibrated! In particular $B(M)$ is a Plateau solution in the spherical quotient $S_{\partial \tilde{M}}/\pi_1(M)$, for the homology class of $S_{\partial \tilde{M}}/\pi_1(M)$ given naturally by $M$.

This embedding is very specific. Questions: Uniqueness? What about non-locally symmetric manifolds?
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The spherical Plateau problem for group homology

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$$S^*(\Gamma) := \{ f \in S(\Gamma); \; \gamma.f \neq f \text{ for any nontrivial } \gamma \in \Gamma \}.$$
A Hilbert model for classifying spaces

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Fact: $S^*(\Gamma)/\Gamma$ is a Hilbert manifold and a classifying space for $\Gamma$. 
Example:

Let \( M \) be a closed oriented \( n \)-manifold. Let \( K := S \ast (\Gamma) / \Gamma \) be as in the previous slide, for \( \Gamma := \pi_1(M) \).

\( M \) determines a class \( h_M \in H^n(K; \mathbb{Z}) \).

Solve the Plateau problem for \( h_M \) and get a Plateau solution \( C^\infty \).

In the special case where \( M \) is hyperbolic, \( S_{\partial \tilde{M}} / \pi_1(M) \) is not isometric to \( K \). In fact, \( M \) does not embed minimally in \( K \).

Nevertheless \( S_{\partial \tilde{M}} / \pi_1(M) \) is isometrically embedded in the ultralimit of \( K \).
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Rm: in the special case where $M$ is hyperbolic, $S_{\partial M}/\pi_1(M)$ is not isometric to $\mathcal{K}$. In fact, $M$ does not embed minimally in $\mathcal{K}$. Nevertheless $S_{\partial \tilde{M}}/\pi_1(M)$ is isometrically embedded in the ultralimit of $\mathcal{K}$. 
Thm: Let \((M, g_{hyp})\) be a closed oriented hyperbolic manifold of dimension \(\geq 3\). Then any Plateau solution for \(h_M\) is intrinsically isometric to \((M, \frac{(n-1)^2}{4n} g_{hyp})\).
Hyperbolic manifolds and intrinsic uniqueness of Plateau solutions

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Conjecture: For \(n = 2\), any Plateau solution for \(h_M\) is intrinsically isometric to an element in the Deligne-Mumford compactification of \(\{\text{hyperbolic metrics on } M\}\).
Let \((M, \frac{(n-1)^2}{4n} g_{hyp}), \Gamma := \pi_1(M)\). Besson-Courtois-Gallot initiated the use of a “barycenter map” \(\text{bar} : S_{\partial \tilde{M}}/\Gamma \to M\).
Let \((M, \frac{(n-1)^2}{4n} g_{hyp}), \Gamma := \pi_1(M)\). Besson-Courtois-Gallot initiated the use of a “barycenter map” \(\text{bar} : S_{\partial\tilde{M}}/\Gamma \to M\).

In our setting, we define a variant 

\[
\text{bar} : \mathcal{K} \to M
\]

such that \(|\text{Jac}_n \text{bar}| \leq 1\) and when \(|\text{Jac}_n \text{bar}|\) is close to 1, the differential \(d\text{bar}\) is almost an isometry.
Let $C_i \subset \mathcal{K}$ be a minimizing sequence “converging” to a Plateau solution $C_\infty$. We are given $\bar{x} : C_i \to M$, maps of degree 1.
About the proof

Let $C_i \subset \mathcal{K}$ be a minimizing sequence “converging” to a Plateau solution $C_\infty$. We are given $\bar{\alpha} : C_i \to M$, maps of degree 1.

Steps:

1. construct a limit map $\bar{\alpha}_\infty : C_\infty \to (M, \frac{(n-1)^2}{4n} g_{hyp})$,
2. $\bar{\alpha}_\infty$ is volume preserving,
3. $\bar{\alpha}_\infty$ is an isometry for the path distances.
Let $C_i \subset K$ be a minimizing sequence “converging” to a Plateau solution $C_\infty$. We are given $\bar{\text{bar}} : C_i \rightarrow M$, maps of degree 1.

Steps:

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Difficulty: lack of a priori regularity for $C_\infty$: need to work on $C_i$ and prove “almost” statements.
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Rm: unlike the hyperbolic case, here there is no good model metric for a barycenter map. We need to work with a sequence of metrics on $M$ approximating the hyperbolic part $M_{hyp}$, and collapsing the rest.
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Sci-Fi question: how much of Geometrization can be recovered with MCF methods?
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Question: Is the Bieberbach embedding stable under MCF?
A general structure result

Let $\Gamma, h \in H_n(\Gamma; \mathbb{Z})$. 

A non-trivial Plateau solution $C^{\infty}$ is never isometrically embedded as a cycle in $K = S^*(\Gamma)$. 

Let $S^{\infty}/\Gamma$ be an "ultralimit" of $K := S^*(\Gamma)/\Gamma$. 

Thm: Any Plateau solution $C^{\infty}$ for $h$ embeds isometrically inside the spherical quotient $S^{\infty}/\Gamma$. Moreover the restriction of $C^{\infty}$ to the smooth part of $S^{\infty}/\Gamma$ is mass-minimizing. 

Let $C^{\infty} > 0 :=$ restriction of $C^{\infty}$ to the smooth part of $S^{\infty}/\Gamma = $ noncollapsed part of $C^{\infty}$. 

The support of $C^{\infty} > 0$ is smooth on a dense open set by Ambrosio-De Lellis-Schmidt.
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Let $\Gamma, h \in H_n(\Gamma; \mathbb{Z})$. A non-trivial Plateau solution $C_\infty$ is never isometrically embedded as a cycle in $\mathcal{K}$... Let $S^\infty/\Gamma_\infty$ be an “ultralimit” of $\mathcal{K} := S^*(\Gamma)/\Gamma$.

Thm: Any Plateau solution $C_\infty$ for $h$ embeds isometrically inside the spherical quotient $S^\infty/\Gamma_\infty$. Moreover the restriction of $C_\infty$ to the smooth part of $S^\infty/\Gamma_\infty$ is mass-minimizing.

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The support of $C^{>0}_\infty$ is smooth on a dense open set by Ambrosio-De Lellis-Schmidt.
A general existence result

Combinatorial properties on $(\Gamma, h) \iff$ Existence of non-trivial mass-minimizing integral currents of the form $C_0^\infty$?
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**Thm**: Let \(\Gamma\) be a torsion-free hyperbolic group with \(h \in H_n(\Gamma; \mathbb{Z}) \setminus \{0\}\) and \(n \geq 2\). Then any Plateau solution \(C_{\infty}\) for \(h\) has a non-empty noncollapsed part \(C_{\infty}^0\).
Combinatorial properties on $(\Gamma, h) \implies$ Existence of non-trivial mass-minimizing integral currents of the form $C_{\infty}^>^0$?

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For instance, $\pi_1$ of negatively curved closed manifolds are torsion-free hyperbolic.