Space of convex ancient solutions to MCF

(work in progress with Angenent and Daskalopoulos)

- \( F: M^n \to \mathbb{R}^{n+1} \)
  \[
  \frac{\partial}{\partial t} F = -H \mathcal{N} \quad \text{(MCF)}
  \]

- \( M^n \) is compact \( \Rightarrow \) finite time singularity \( T \).

- Study finite time singularities through a blow-up procedure which yields as limits ancient solutions.

**def.** Ancient solutions to MCF are solutions to (MCF) that exist for all \( t \in (-\infty, a) \), \( 0 \leq a \leq +\infty \). If \( a = +\infty \) we call them eternal solutions.

**Huisken:** \( M^n \to \mathbb{R}^{n+1} \) compact, convex hypersurface. The (MCF) starting at \( M^n \) has a smooth solution until finite time at which the flow extincts to a point. The rescaled MCF converges exponentially as \( t \to +\infty \) to a round sphere \( S^n \).
Examples of ancient solutions

- **Solitons** (self-similar): $S^k \times R^{n-k}$, translating solitons (Bowl soliton)
- Examples that are **not** solitons are discovered by White; Haslhofer-Hershkovits. They have $O(j+1) \times O(n-j)$ symmetry, asymptotic to $S^j \times R^{n-j}$ as $t \to -\infty$.

![Cylinder and Bowl Soliton Diagram]

**Def.** (Sheng, Wang): $M^n$ is $\mathcal{L}$-noncollapsed if

$\exists r > 0$ both of radius at least $\frac{r}{H(p)}$, tangent to $M^n$ at $p$.

**Andrews:** $\mathcal{L}$-noncollapsed condition is preserved along the flow.

**Def.** Ancient **oval**: compact, $\mathcal{L}$-noncollapsed ancient MCF solution, not self-similar.
Brendle, Choi: Let $M^t$, $t \in (-\infty, 0)$ be noncompact, strictly convex, uniformly $2$-convex $(\lambda_1 + \lambda_2 \geq \beta H)$, and noncollapsed ancient MCF solution in $\mathbb{R}^{n+1}$. It is the Bowl soliton.

Angenent - Daskalopoulos - S.: Let $M^t$ be uniformly $2$-convex Ancient oval. Then, up to ambient isometries, translations and parabolic rescaling, it is the Ancient oval constructed by White; Haslhofer - Hershkovits.

Choi, Haslhofer, Hershkovits, White: Obtained the classification from above assuming the parabolic blow down is a round cylinder. As a corollary they showed the flow through cylindrical and sphenical singularities is unique. They also show the Mean convex neighborhood conjecture.
Du, Haslhofer: Any nontrivial $O(k) \times O(n+1-k)$ symmetric ancient noncollapsed ancient MCF solution is up to scaling and time shift the one constructed by White; Haslhofer-Haslhofer-Shkolovits.

- Tangent flow at time $-\infty$ is $\mathbb{R}^k \times S^{n-k}(\sqrt{2(n-k)t})$.

- They also show for $k \geq 2$, there exists a $(k-1)$-parameter family of distinct ancient ovals that are only $O(n+1-k)$ symmetric.

Question:
Facts about convex sets:

- $M = \hat{\mathcal{M}}$, $\hat{\mathcal{M}}$ - convex. Then:
  i) either $M = \mathbb{R}^n$
  ii) or $M = \mathbb{R}^k \times N$, $0 \leq k < n$ and $N = \mathcal{N}$, $\mathcal{N}$ is closed convex set with interior that has no infinite line. $N$ is homeomorphic to $S^{n-k}$ or $\mathbb{R}^{n-k}$.

- If $M = \mathcal{M}$ as above and $M$ homeomorphic to $\mathbb{R}^n$ containing no infinite line, assume $\{x_{n+1} = 0\}$ is a supporting plane to $\hat{M}$ at the origin, then $D = \overline{T}(\hat{M})$ where $\overline{T} : \mathbb{R}^{n+1} \rightarrow \{x_{n+1} = 0\}$ is the standard orthogonal projection.

- $D$ is called the shadow of $M$. 
• [Existence] and [uniqueness] of MCF starting at $\mathcal{C}$, where $\mathcal{C} \subset \mathbb{R}^{n+1}$ is a convex set.

**Existence:** $\mathcal{C} \⊂ \mathbb{R}^{n+1}$. Define its evolution by MCF to be

$$C_t = \overline{U D_t}$$

over all smooth compact solutions $\{D_t : 0 \leq t < S\}$ to MCF with

$D_0 \subset \text{Int } C_0$, \quad C_0 = C$.

• If $C_0$ is compact and has smooth boundary $\Rightarrow C_t = C_t^\mu$. 

Lemma: If $C_{k,t}, 0 \leq t < T_k$ is a sequence of compact evolutions to MCF with $C_{k,0} \subset C_{k+1,0}$, then

a) $T_k \leq T_{k+1}$

b) $C_{k+1,t} \subset C_{k+1,t}, t \in [0, T_k)$

c) $C_{0,0} := \bigcup_{k, t < T_k} C_{k,t}$ is the MCF evolution of $C_{0,0}$ and defined for $t < T = \sup T_k$.

Regularity of MCF constructed above

Claim: $\partial C_t$ is smooth for each $t > 0$ and $\{\partial C_t \mid 0 < t < T \}$ is a smooth MCF solution.
**Uniqueness:** Let $M_t^1, M_t^2 \in C^1(\mathbb{R}^{n+1})$, $t \in (0,T]$ be two smooth convex solutions to MCF with

$$\lim_{t \to 0} M_t^1 = \lim_{t \to 0} M_t^2 = M_0.$$ Then

$$M_t^1 = M_t^2 \quad \forall t \in (0,T].$$

**Proof:**
Proof:

Theorem: There is no convex $M_0$, $E!$ smooth solution to MCF for $t \in (0, T]$. 
Want to define a notion of being \( \mathcal{L} \)-noncollapsed for possibly noncompact, nonsmooth surfaces.

**Definition** \( \mathcal{M} \subseteq \mathcal{C} \mathcal{R}^{n+1} \) complete, convex. We say \( \mathcal{M} \) is \( \mathcal{L} \)-noncollapsed if \( \mathcal{M} = \bigcup_{k=1}^{\infty} \mathcal{C}_k \), \( \{\mathcal{C}_k\} \) is an increasing sequence of smooth convex sets so that each \( \mathcal{C}_k \) is \( \mathcal{L}_k \)-noncollapsed and the limit \( \lim_{k \to \infty} \mathcal{L}_k = \mathcal{L} \).

**Lemma** \( \mathcal{C} \subseteq \mathcal{C} \mathcal{R}^{n+1} \) complete, convex so that \( \partial \mathcal{C} \) is \( \mathcal{L} \)-noncollapsed. Let \( C_t \) be a unique smooth MCF coming out of \( C \), \( t \in [0,T) \). Then \( \partial C_t \) is also \( \mathcal{L} \)-noncollapsed for \( t \in (0,T) \).
We put weak topology on the space of convex sets.

\[ X = \{ C \subset \mathbb{R}^{n+1}, \text{convex with } C \neq \emptyset \} \]

\[ \forall C \in X \text{ define the Huisken measure } \]

\[ d\mu_C = \left( \frac{4\pi}{n} \right)^{n/2} e^{-\frac{1}{4} x^2} d\mathcal{H}^n \]

\[ n\text{-dim Hausdorff measure on } C \]

where

\[ \mathcal{H}(C) = \frac{1}{\left( \frac{4\pi}{n} \right)^{n/2}} \int_C e^{-\frac{1}{4} x^2} d\mathcal{H}^n \]

\[ d\mu_C \text{ is given by: } \forall f \in C_0^\infty(\mathbb{R}^{n+1}) \]

\[ \langle \mu_C, f \rangle = \lim_{\varepsilon \downarrow 0} \frac{1}{\varepsilon} \int_{B_\varepsilon(C) \setminus C} e^{-\frac{1}{4} x^2 / \varepsilon} f(x) \, dx \]

\[ \text{Identify } \]

\[ X = \{ \mu_C \mid C \subset \mathbb{R}^{n+1}, \text{convex, } C \neq \emptyset \} \]
If $C$ is the space of bounded continuous functions $f : \mathbb{R}^{n+1} \to \mathbb{R}$, $\lim_{\|x\| \to \infty} f(x)$, any measure $\mu$ defines a linear functional on $C$ by

$$\langle \mu, f \rangle = \int f \, d\mu$$

Regard $X$ as a subspace of $C^\ast$.

Topology on $X$ is the weak* topology on $C^\ast$.

\[ \text{def} \quad C_k \subset \mathbb{R}^{n+1} \quad \text{convex}, \quad C_k \neq \emptyset \]

We say $C_k \to C \ (k \to \infty)$ in the strong sense if $d(c_k(x), c(x)) \to 0$ as $k \to \infty$, for all $x \in \mathbb{R}^{n+1}$, where $C_k, C \in X$ and $d_c(x) = d(x, C)$.

**Lemma:** If $M_{C_k} \to M_c$ as $k \to \infty$ in the weak sense, where $C_k, C \in X$, then $d(x, C_k) \to d(x, C)$, $\forall x \in \mathbb{R}^{n+1}$. 
Define for $0 < h_0 < h_1 < 2$

\[ X_\pm(h_0, h_1) = \{ C \subseteq \mathbb{R}^{n+1} \mid \text{convex, } \hat{C} \neq \emptyset, \forall C \text{ is } \mathcal{L}\text{-noncollapsed}, \ h_0 \leq \mathcal{H}(C) \leq h_1 \} \]

**Lemma:** \( X_\pm(h_0, h_1) \) is compact.

Regard the RMCF

\[ \frac{\partial}{\partial t} F = -\Delta V + \frac{1}{2} F^2 \]  \hspace{1cm} \text{(RMCF)}

as a flow on \( X \).

**Fixed points of the flow:** \( S^k_{\bar{c}_k} \times \mathbb{R} \)

\[ 1 = \mathcal{H}(\mathbb{R}^k) < \sqrt{2} < \ldots < \mathcal{H}(S^{k+1}) < \mathcal{H}(S^k) < \ldots < \mathcal{H}(S^1) < 2 \]

(RMCF) defines a map

\[ \phi : D \rightarrow X \]

\[ D = \{ (M_c, t) \in X \times [0, +\infty) \mid M_c \in X, 0 \leq t \leq T_c \} \]

\[ \phi(M_c, t) = \phi^t(M_c) \]
We show $\phi$ is a continuous flow on $D$.

Define for $0 < h_0 < h_1 < 2$

$$ I(\lambda, h_0, h_1) = \{ C \in X \mid \exists C \text{ is } \lambda \text{-non-collapsed, for which } \exists \text{ an entire solution } \{ C_t \}_{t \in \mathbb{R}} \text{ of RMCF with } C_0 = C \text{ and } h_0 < \mathcal{H}(C_t) < h_1 \} $$

**Proposition:** For $0 < \lambda < \lambda^*$ and $0 < h_0 < h_1 < 2$, $h_i \notin \mathcal{H}(\Sigma^k)$ for all $0 < k < n$. Then

(i) $I(\lambda, h_0, h_1)$ is compact

(ii) $I(\lambda, h_0, h_1)$ consists of all fixed points $\Sigma^k$ with $h_0 < \mathcal{H}(\Sigma^k) < h_1$, all hyperplanes through the origin and all connecting orbits between them.
Conjecture: The set \( I( \lambda, \nu_0, \nu_1) \)
containing fixed points \( \Sigma^k \) is
homeomorphic to an \((n-1)\)-dimensional simplex.

From now on we consider RMCF defined on
\[
\mathcal{Z}_S(\lambda) = \{ C \in Y_S(\lambda) \mid \tau(C) = 1 \}
\]
\[
Y_S(\lambda) = X_S(\lambda) / SO(n+1)
\]
\[
X_S(\lambda) = \{ C \in X \mid \exists C \text{ is } \lambda\text{-noncollapsed},
\quad C = -C \text{ (invariant under point reflection)} \}
\]
where \( \tau(C) \) is the singular time for the (MCF).
Remark. $C \in X_s(L)$ is either compact or is of the form $\tilde{C} \times \mathbb{R}^k$ for some compact symmetric space $\tilde{C}$.

1. $\mathcal{F}_s(L, h_0, h_1) = \mathcal{F}(L, h_0, h_1) \cap X_s(L)$

\[
\begin{array}{ccc}
\mathbb{R}^3 & \xrightarrow{\text{unique orbit by our result}} & \mathbb{S}^2 \\
\text{In} & & \text{(ADS)}
\end{array}
\]

\[
\begin{array}{ccc}
S^1 \times \mathbb{R} & \longrightarrow & \mathbb{S}^2 \\
\text{In} & & \\
\mathbb{R}^4 & & \text{(ADS)}
\end{array}
\]

\[
\sum^1 = S^1_{\frac{1}{12}} \times \mathbb{R}^2, \quad \sum^2 = S^2_{\frac{1}{48}} \times \mathbb{R}, \quad \sum^3 = S^3_{\frac{1}{6}}
\]

2. There are connecting orbits from $\sum^i$ to $\sum^d$ whenever $i < d$.

\{ADS\}: The orbits from $\sum^1$ to $\sum^2$ and from $\sum^2$ to $\sum^3$ are unique.
Du-Haslhofer

(a) There is a 1-parameter family of orbits \( \Sigma^1 \rightarrow \Sigma^3 \) that are \( O(2) \)-symmetric.

(b) There is an orbit which is \( O(2) \times O(2) \)-symmetric and it is unique.

\[ \Sigma^1 \quad \Gamma(1,2) \quad \Sigma^2 \quad \Gamma(2,3) \quad \Sigma^3 \] in \( \mathbb{R}^4 \)

**Theorem:** Let \( \mathcal{W} \subset \mathcal{E}_s(z) \) be any open neighborhood of \( \Gamma(1,2) \cup \Gamma(2,3) \). Then \( \mathcal{W} \) contains a connecting orbit from \( \Sigma^1 \) to \( \Sigma^2 \).
Proof:

- $U_2$ is an open set in $\mathcal{B}_*(\Delta)$
- $K \subset U_2$ is compact

Step 1: