Boundary regularity of area-minimizing currents: a linear model with analytic interface

Zihui Zhao
joint work with Camillo De Lellis

University of Chicago

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Plateau’s problem: Given a closed curve $\Gamma$, what is the surface $T$ that spans $\Gamma$ with the least area?

Figure: Soap film. Photo credit: archdaily.com
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Depending on what class of surfaces we are minimizing, there are different formulations of Plateau’s problem:

- Classical area-minimizing surfaces using the parametric approach (as images of a disk, or surfaces of higher genus $\Sigma_g$), Douglas, Radó, Courant et al.
- Integral currents, Federer-Fleming et al.
- Set-theoretic approach, Reifenberg, Harrison-Pugh, David et al.
- Integral varifolds, Almgren, Allard et al.
Integral currents: model orientable submanifolds

In 1960 Federer and Fleming introduced the notion of integral current and proved the existence of area-minimizers in this class.

Definition

We say a current $T \subset \mathbb{R}^{m+n}$ of dimension $m$ is integer rectifiable, if there are

- countably $m$-dimensional orientable $C^1$ submanifolds $M_i \subset \mathbb{R}^{m+n}$,
- pairwise disjoint closed sets $A_i \subset M_i$,
- positive integers $k_i \in \mathbb{N}$,

such that

$$T = \sum_i k_i [A_i], \quad \text{modulo a set of zero } \mathcal{H}^m\text{-measure}.$$ 

We say a current $T$ is integral if both $T$ and its boundary $\partial T$ are integer rectifiable.
Area-minimizing current

Definition (Area-minimizing current)

Let $T$ be an $m$-dimensional integral current in $\mathbb{R}^{m+n}$ (or in a Riemannian manifold $M^{m+n}$). We say $T$ is area minimizing, if

$$\text{Area}(T') \geq \text{Area}(T)$$

for any competitor $T'$ of $T$, that is, $T'$ is an integral current such that $T' = T$ outside of some compact set.
Interior regularity: codimension one ($n = 1$)

Theorem (De Giorgi, Simons, Federer, Simon et al.)

Assume $T$ is an area-minimizing current and $n = 1$. Then

1. If $2 \leq m \leq 6$, $T$ is regular, i.e. $\text{Sing}_i(T) = \emptyset$.
2. If $m = 7$, $\text{Sing}_i(T)$ consists of isolated points.
3. If $m > 7$, $\text{Sing}_i(T)$ has Hausdorff dimension at most $m - 7$, and it is $(m - 7)$-rectifiable.
When the codimension $n \geq 2$, branch point starts to emerge.

**Example of branch singularity.** The holomorphic curve

$$C := \{(z, w) \in \mathbb{C}^2 : z^2 = w^3\}$$

is a 2-dimensional area-minimizing current in $\mathbb{R}^4$. The origin is singular despite that $C$ has a flat tangent plane $\{(z, w) \in \mathbb{C}^2 : z = 0\}$ at the origin.
Theorem (Almgren, Chang, DeLellis-Spadaro, DeLellis-Spadaro-Spolaor)

Assume $T$ is an area-minimizing current and $n \geq 2$. Then

1. When $m = 2$, $\text{Sing}_i(T)$ consists of isolated points.
2. $\text{Sing}_i(T)$ has Hausdorff dimension at most $m - 2$. 
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Remark

- In particular, if $T$ is a two-dimensional area-minimizing current, then locally $\text{spt}(T)$ is a branched minimal surface.
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- By a dimension reduction argument, it suffices to study flat singular points.
Almgren’s proof of interior regularity

Near a flat singular point $p$, approximate the current $T$ locally by the graph of a multi-valued function

$$f : \mathbb{D} \subset \mathbb{R}^m \to \mathcal{A}_Q(\mathbb{R}^n)$$

where $\mathcal{A}_Q(\mathbb{R}^n)$ is the metric space of unordered $Q$-tuples of points in $\mathbb{R}^n$, and the integer $Q$ equals $\Theta(T, p) := \lim_{s \to 0} \frac{\|T\|(B_s(p))}{\omega_m s^m}$. 
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2. If the current $T$ is area-minimizing, $f$ is close to be a minimizer of the Dirichlet energy $\text{Dir}(f, \mathbb{D}) := \int_{\mathbb{D}} |Df|^2$. 
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The reason is that

$$\|T\|(C_r(x)) - Q \cdot \omega_m r^m \approx \frac{1}{2} \int_{B_r(x)} |Df|^2,$$

where $C_r(x)$ denotes the cylinder $B_r(x) \times \mathbb{R}^n \subset \mathbb{D} \times \mathbb{R}^n$. 

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Prove analogous regularity result for multi-valued functions which minimize the Dirichlet energy.
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4. Prove analogous regularity result for multi-valued functions which minimize the Dirichlet energy.
On the boundary, a regular point can be either *one-sided* or *two-sided*.

**Example.** Let $\pi$ be a two-dimensional plane and $\Gamma = \Gamma_1 \cup \Gamma_2$. The area-minimizing current $T$ which bounds $\Gamma$ is the sum of the two disks bounded by $\Gamma_1$ and $\Gamma_2$, counting multiplicity.
The first boundary regularity result is by Allard (for varifolds):

**Theorem (Allard 1969)**

1. If \( p \in \Gamma \) is a point where the density \( \Theta(T, p) \) equals \( \frac{1}{2} \), i.e. \( p \) is one-sided, then \( p \in \text{Reg}_b(T) \).

2. If there is some wedge \( W \) of opening angle smaller than \( \pi \) whose tip contains \( p \) and such that \( \text{spt}(T) \subset W \), then \( \Theta(T, p) = \frac{1}{2} \) and thus \( p \in \text{Reg}_b(T) \).
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**Remark**

DeLellis-Nardulli-Steinbrüchel: When \( \partial T = Q[\Gamma] \), any boundary point \( p \in \Gamma \) with density \( < \frac{Q+1}{2} \) is regular.
Boundary regularity in codimension one

Theorem (Hardt-Simon 1979)

Let $\Gamma \subset \mathbb{R}^{m+1}$ be a $C^{1,\alpha}$ closed oriented embedded submanifold of dimension $m - 1$. Suppose $T$ is an area-minimizing current with boundary $\Gamma$, then $\text{Sing}_b(T) = \emptyset$.

Remark

In particular when $m = 2$, $\text{spt}(T)$ is an embedded surface (with boundary) of finite genus.
Again, the case of higher codimensions is different.

- **Genuine branch singularity.** For example, cut the minimizing current \( \{(z, w) \in \mathbb{C}^2 : z^3 = w^{3k+1}\} \) where \( k \in \mathbb{N} \).
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- **Self-intersection, or singular point of **crossing** type.**
  - For example, let \( \pi_1, \pi_2 \subset \mathbb{R}^4 \) be two-dimensional planes such that \( \pi_1 \cap \pi_2 = \{0\} \). Then \( T = [\pi_1^+] + [\pi_2] \) is an area-minimizing current with boundary \( \mathbb{R} \), and \( 0 \in \text{Sing}_b(T) \).
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  - Alternatively, add to \( \{(z, w) \in \mathbb{C}^2 : z^2 = w^3\} \) a half plane \( [\pi^+] \), where \( \pi \cap \mathbb{C}^2 = \{0\} \).
Until recently it is not even known if $\text{Reg}_b(\Gamma) \neq \emptyset$ for general, non-convex boundary $\Gamma$. 

**Theorem (DeLellis-DePhilippis-Hirsch-Massaccesi)**

Let $\Gamma \subset \mathbb{R}^m + n$ be a $C^3, \alpha$ closed oriented submanifold of dimension $m - 1$. Suppose $T$ is an area-minimizing current with boundary $\Gamma$, then $\text{Reg}_b(T)$ is open and dense in $\Gamma$.

**Remark**

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Boundary regularity in higher codimensions (cont.)

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Boundary singularity (when $m = 2, \ n \geq 2$)

**Question**: Can we analyze the size of the boundary singular set, in the particular case of two-dimensional area-minimizing currents?
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**Conjecture**

*When $\Gamma$ is a closed analytic curve, and $T$ is an area-minimizing current with $\partial T = [\Gamma]$, then $\text{Sing}_b(T)$ is discrete.*
Analytic boundary: motivations

Theorem (Gulliver-Lesley 1973, Gulliver 1977, White 1997)

Let $\Gamma$ be a closed analytic curve, and let $T$ be a classical area-minimizing surface spanning $\Gamma$. Then $T$ has no boundary branch point.
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There are a $C^\infty$ simple closed curve $\Gamma \subset \mathbb{R}^4$ and an area-minimizing current $T$ spanning $\Gamma$, such that $\text{Sing}_b(T)$ has an accumulation point.
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Remark

This is due to the failure of unique continuation at the boundary.
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Assume \( \Gamma \) is an analytic curve in \( \mathbb{R}^{2+n} \). We write \( \Gamma = (\gamma, \varphi) \), where \( \gamma \) is the projection of \( \Gamma \) onto a two-dimensional plane \( \pi \) and \( \varphi \in \pi^\perp \).
Analytic boundary: setup of the linearized model

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**Definition**

We say that a pair $f = (f^+, f^-)$ is in the space $W^{1,2}(\mathcal{D}, \mathcal{A}^\pm_Q)$ with interface $(\gamma, \varphi)$, if

$$f^+ \in W^{1,2}(\mathcal{D}^+, \mathcal{A}_{Q+1}) \text{ and } f^- \in W^{1,2}(\mathcal{D}^-, \mathcal{A}_Q);$$
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1. $f^+ \in W^{1,2}(\mathbb{D}^+, \mathcal{A}_{Q+1})$ and $f^- \in W^{1,2}(\mathbb{D}^-, \mathcal{A}_Q)$;
2. $f^+|_\gamma = f^-|_\gamma + [\varphi]$. 

Dirichlet energy-minimizers with analytic interface

**Theorem (DeLellis-Z.)**

*Given an analytic interface $(\gamma, \varphi)$, suppose $f \in W^{1,2}(\mathbb{D}, A_{Q}^\pm)$ minimizes the Dirichlet energy among all competitors with the prescribed interface. Then the singular set of $f$ is discrete.*
Dirichlet energy-minimizers with analytic interface

Theorem (DeLellis-Z.)

Given an analytic interface \((\gamma, \phi)\), suppose \(f \in W^{1,2}(D, A^\pm_Q)\) minimizes the Dirichlet energy among all competitors with the prescribed interface. Then the singular set of \(f\) is discrete.

Remark

Exceptional case: non-homogeneous blow-down of half of the Enneper surface.
Thank you!