LIMIT LINEAR SERIES, THE IRRATIONALITY OF $M_g$, 
AND OTHER APPLICATIONS

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ABSTRACT. We describe degenerations and smoothings of linear series 
on some reducible algebraic curves. Applications include a proof that 
the moduli space of curves of genus $g$ has general type for all $g \geq 24$, a 
proof that the monodromy action is transitive on the set of linear series 
of dimension $r$ and degree $d$ on a general curve of genus $g$ when $\rho := 
g - (r+1)(g-d+r) = 0$, a proof that there exist Weierstrass points with 
every semigroup of a certain class—in particular, on curves of genus 
g, all those semigroups with weight $w \leq g/2$ occur and a proof that 
the monodromy group acts as the full symmetric group on the $g^3 - g$ 
Weierstrass points of the general curve.

Curves will here be reduced, connected, and complex algebraic.

The study of general curves (Brill-Noether theory, etc.) and of moduli 
of curves depends on the degeneration of smooth curves to singular ones. 
Originally, the singular curves used were irreducible curves with nodes ([G-H] 
is a recent avatar) or, more recently, cusps [E-H1], but from the work of 
Mumford and others on the moduli space of stable curves it is apparent that 
reducible curves should be considered as well.

Unfortunately the degeneration of a linear series on a curve which degener-
ates to a reducible curve has not been well understood except in the par-
ticularly simple case of pencils; there the “limit” of the linear series, after 
removing base points, corresponds to an admissible covering, in the sense of 
Beauville, Knudsen and Harris-Mumford [B, K, H-M], of a curve of genus 
0. The potential of a general theory is indicated, for example, by work of 
Gieseker [G].

In this announcement we describe the limits of linear series on some reduc-
able curves and give some applications.

We call a curve tree-like if its irreducible components meet only two at a 
time, in ordinary nodes, in such a way that its dual graph (a vertex for each 
component, an edge for each intersection between distinct components) has 
no loops.

We say that a curve is of compact type if its (generalized) Jacobian is 
compact, or, equivalently, if it is tree-like and its irreducible components are 
all nonsingular.

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DEFINITION. A limit \( g_d^* \) on a tree-like curve \( Y \) is a collection of \( g_d^* \)'s, one on each irreducible component \( Z \) of \( Y \),

\[
L_Z \text{ a line bundle of degree } d \text{ on } Z,
\]

\[
V_Z \subset H^0(Z, L_Z) \text{ an } r + 1 \text{-dimensional subspace}
\]
such that whenever two components of \( Y \) meet in a point, say \( p = Z_1 \cap Z_2 \), there is for each \( \sigma \in V_{Z_1} \) a \( \tau \in V_{Z_2} \) such that \( \text{ord}_p \sigma + \text{ord}_p \tau = d \).

The following result is implicit in [E-H3]:

**Theorem 1.** Let \( \mathcal{O} \) be a discrete valuation ring, and let \( X \to \text{Spec } \mathcal{O} \) be a family of curves with irreducible geometric general fiber \( X_{/\mathcal{O}} \) and reduced, special fiber of compact type. Given a line bundle \( L \) and a \( g_d^* k(\eta)^{r+1} \equiv V \subset H^0(X_{/\mathcal{O}}, L) \) on \( X_{/\mathcal{O}} \), there is a family \( X' \to \text{Spec } \mathcal{O}' \) obtained from \( X \) by base change, blow-ups of points in the central fiber, and normalizations, with reduced, special fiber \( Y' \) of compact type such that:

1. For each irreducible component \( Z \subset Y \) there is an extension \( L_Z \) of \( L \) to \( X \) with
   \[
   \deg(L |_{Z}) = d,
   \]
   \[
   \deg(L |_{Z'}) = 0 \text{ for irreducible components } Z' \neq Z.
   \]
2. The images
   \[
   V_Z = \text{im}(V \hookrightarrow \pi'_*(L_Z)^{\text{restriction}} \to H^0(Z, L |_{Z'}))
   \]
form a limit \( g_d^* \) on \( Y \).

See [E-H2,3] for applications of this result to Brill-Noether theory.

We will say that a limit \( g_d^* \) on a tree-like curve \( Y \) is smoothable if it can be obtained from a family with geometrically irreducible general fiber as in Theorem 1. Every limit \( g_d^* \) is smoothable, as is shown in [H-M]; an explicit analytic smoothing can actually be constructed with little effort. Unfortunately there are nonsmoothable \( g_d^* \)'s with \( r \geq 2 \). But these only occur on rather atypical curves, as our next result shows:

**Theorem 2.** Let \( X \to B \) be a family of tree-like curves of arithmetic genus \( g \) over an irreducible base \( B \), and let \( G_d^*(X/B) \) be the corresponding family of limit \( g_d^* \)'s. Set \( \rho = g - (r + 1)(g - d + r) \). (\( \rho \) may be negative!) If \( \text{dim } G_d^*(X/B) \leq \text{dim } B + \rho \), then every limit \( g_d^* \) on every curve of the family is smoothable.

Curves satisfying the hypothesis of Theorem 2 (with \( B \) a point) may be found in [E-H2,3]. It is also satisfied (for every \( r, d \)) by the union of three general curves of genus \( g_1, g_2 \) with \( g_1 + g_2 = g \), joined at general points of each, and by many other simple curves and families of curves.

Theorem 2 is proved by giving explicitly the "right number" of local equations for the family of \( g_d^* \)'s (or rather, for a certain associated frame-bundle) in the neighborhood of a given limit \( g_d^* \). This approach was suggested by conversations with Ziv Ran, to whom we are grateful.

We now indicate three applications beyond those of [E-H2,3]:

First, we may complete and simplify the ideas in the second half of [H-M] and [H], where it is shown that the moduli space \( M_g \) of curves of genus \( g \) has general type for \( g \) odd and \( g \geq 25 \) or even and \( g \geq 40 \):
APPLICATION 1 \[E\text{-H5}\]. \(M_g\) has general type for all \(g \geq 24\).

For the proof of this we make use of the ideas and methods of the first 3 sections of \[H\text{-M}\] as described in the introduction to \[H\]; these methods require the choice and computation of a divisor in \(M_g\) with certain properties.

We distinguish 2 (overlapping) cases:

(i) If \(g + 1\) is not prime, then for suitable \(r\) and \(d\) we have

\[
\rho = g - (r + 1)(g - d + r) = -1,
\]

and the closure of the set of smooth curves possessing a \(g_d^r\) forms a suitable divisor in \(M_g\) if \(g \geq 24\). This covers in particular the cases \(g\) odd and \(g = 24, 26\).

(ii) If \(g\) is even, say \(g = 2k - 2\), and \(g \geq 28\), we use the closure of the ramification divisor of the map from the moduli space of curves \(C\) of genus \(g\) with chosen pencil \(G^2 \cong V \subset H^0(C, \mathcal{L})\) of degree \(k\) to \(M_g\), in accordance with the program expressed in the introduction to \[H\text{-M}\]. To circumvent the problem mentioned in the introduction \[H\] we interpret ramification as being signalled by the presence of a nonzero section of \(K_C \otimes \mathcal{L}^{-2}\), where \(K_C\) is the canonical class of \(C\).

As a second application, we can complete, in a certain sense, the result of Fulton and Lazarsfeld \[F\text{-L}\] who prove (using the result of Gieseker proved in \[G\] and \[E\text{-H3}\]) that if \(C\) is a general curve, then the variety \(G^r_d(C)\) of \(g_d^r\)'s on \(C\) is irreducible as long as \(\rho := g - (r + 1)(g - d + r) > 0\). For \(\rho = 0\) and \(C\) general, \(G^r_d(C)\) is a reduced set of points. We prove:

APPLICATION 2 \[E\text{-H4}\]. Assume \(\rho = g - (r + 1)(g - d + r) = 0\). The fundamental group of the moduli space of curves \(C\) with \(G^r_d(C)\) reduced and finite acts transitively by monodromy on each such \(G^r_d(C)\). Equivalently, there is a family of such curves \(X \to B\) such that the associated family \(G^r_d(X/B)\) is irreducible.

The key to the proof of this is the fact that on the curve used in \[E\text{-H3}\] the different \(g_d^r\)'s can be labelled, in the \(\rho = 0\) case, by certain chains of Schubert cycles in a Grassmann variety. Further, if two of these chains differ in only one element, then a family of curves can be constructed (by allowing two “elliptic tails” to hang at varying points from one rational component of a curve as in \[E\text{-H3}\]) whose monodromy interchanges the corresponding \(g_d^r\)'s. Since the simplicial complex of chains of Schubert cycles is connected in codimension 1 (even Cohen-Macaulay—see for example \[D\text{-E\text{-P}}\]), this suffices to prove transitivity.

APPLICATION 3. Certain semigroups occur as the Weierstrass semigroups of smooth curves. In particular, if \(\Gamma = \{0, a_1, a_2, \ldots\} \subset \mathbb{N}\) is a subsemigroup without common divisor of the natural numbers, then \(\Gamma\) occurs as the Weierstrass semigroup of a curve of genus \(g = |\mathbb{N} - \Gamma|\) if \(a_1 > w\) or, more particularly, \(w \leq g/2\), where \(w = \sum_{i=1}^{\ell+1} (g + i - a_i)\) is the weight of \(\Gamma\). Moreover, there is at least one component of the subvariety of Weierstrass points with semigroup \(\Gamma\), in \(M^1_g\), with codimension \(w\).

This is proved inductively by smoothing “limit canonical series” on curves of the form
where $C$ is a curve of genus $g - 1$ with a suitable Weierstrass point $p$ of a certain type, moving in a family whose dimension is the weight of $p$, $E$ is an elliptic curve, $q-p$ is torsion of a suitable order, and the limit series is chosen to have ramification at $q$ corresponding to a Weierstrass point of the desired type.

**Application 4.** The monodromy group acts on the $g^3 - g$ Weierstrass points of a general curve as the symmetric group on $g^3 - g$ letters.

This is proved by specializing to a reducible curve with a positive dimensional family of “limit canonical series”, and examining the monodromy of this family.

**Remark.** It seems possible to give a related, but substantially more complicated, description of “limit $g^r$’s” on arbitrary stable curves. It may be possible to use this fact to study other types of Weierstrass points of low weight.

**References**


[E-H5] ———, $M_g$ is of general type for $g \geq 24$ (in preparation).


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