# OPEN QUESTIONS IN ALGEBRAIC GEOMETRY

These are the notes of a discussion on open problems that was held at MSRI in March 2009. We have included the name of the person who proposed each question. These notes are an unofficial record of this discussion, and we make no guarantees for the correctness of the various claims. If you find a mistake in the notes, please email Daniel Erman (derman at math.berkeley.edu).

## 1. Numerically $f$-trivial divisors (Kollár)

**Background.** $(X, \Delta)$ is log canonical if $K_X + \Delta$ is $\mathbb{Q}$-Cartier and for all log resolutions, $f : Y \to X$, there is a a $\mathbb{Q}$-Cartier divisor $\Delta_Y \subseteq Y$ satisfying:

- $f_*\Delta_Y = \Delta$
- $K_Y + \Delta_Y \sim_{\mathbb{Q}} f^*(K_X + \Delta)$ (The equivalence is that some scalar multiple of the left hand side equals the right hand side.)
- coefficients of $\Delta_Y \leq 1$.

**Question 1.1.** *Let $Y$ smooth, $f : Y \to X$ and $f_*\mathcal{O}_Y = \mathcal{O}_X$. Further let there exist a $\mathbb{Q}$-divisor $\Delta_Y$ such that have $K_Y + \Delta_Y$ is numerically $f$-trivial, and such that the coefficients of $\Delta_Y \leq 1$. Further, assume either $\Delta_Y$ is normal crossing or that $(Y, \Delta_Y)$ is log canonical.*

*Then is $K_X + f_*\Delta_Y$ $\mathbb{Q}$-Cartier? Equivalently, is $K_Y + \Delta_Y$ $f$-trivial?*

This question asks for an extension of a result of Kawamata: if $X$ is the cone over $Z$ and $K_Z \equiv_{num} 0$ and $\pi : Y \to X$ is the blowup and $\Delta_Y$ is the exceptional divisor. Assume $Z$ is smooth or LC. Then it is known that $K_Z \sim_{\mathbb{Q}} 0$.

Here are some possible reduction steps:

(1) Some kind of MMP should reduce this to the case when $\Delta_Y$ equals the reduced exceptional divisor of $f$ and $(Y, \Delta_Y)$ is dlt.
(2) If we knew semi log abundance (applied to $\Delta_Y$), our problem should follow. However, we have a simpler case here. In the usual proofs of (log) abundance, the hard part is to construct a morphism which is a candidate for the pluricanonical map. Here we have the map already constructed for us. So this should be much easier.
(3) The surface case has been known for some time, mostly as a result of the classification of lc surface singularities. Dimension three follows from abundance. Hence, the first open case would be when $X$ has dimension four.

## 2. Diophantine subsets of functions fields of curves (Kollár)

**Background.** This is related to the study of diophantine sets over function fields, as in Kollár's "Diophantine subsets of function fields of curves". Let $X$ be an affine variety with a dominant map $X \to \mathbb{A}^1_t$. For a rational section $s : \mathbb{A}^1_t \dashrightarrow X$, we measure its "poles" as follows. Fix another morphism $p : X \to \mathbb{A}^1_x$. Then $p \circ s : \mathbb{A}^1_t \to \mathbb{A}^1_x$, so it is a rational function in $t$.

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**Question 2.1.** *Is there some $X \to \mathbb{A}^1_t$ such that:*

- *every rational section $s : \mathbb{A}^1 \to X$ gives a polynomial $p \circ s : \mathbb{A}^1_t \to \mathbb{A}^1_x$, and*
- *the degrees of these polynomials is unbounded?*

**Question 2.2.** *Is there some $X \to \mathbb{A}^1_t$ defined over $\mathbb{R}$ such that*

- *every rational section $s : \mathbb{A}^1 \to X$ gives a rational function that is defined over $\mathbb{R}$ gives $p \circ s : \mathbb{A}^1_t \to \mathbb{A}^1_x$ without real poles, and*
- *the order of the poles at infinity is unbounded?*

One would hope that there is such an $X$. Then Question 2 should be much easier. This is the case that is most of interest for Diophantine sets.

## 3. Minimal degree lifting of generic point of $\mathcal{A}_g$ (Vistoli)

**Background.** Consider $A_g$ the moduli space of principally polarized abelian varieties of dimension $g$. Does there exist a principally polarized abelian $X$ over the function field $K(A_g)$?

For some finite extension $L \supseteq K(A_g)$, we have the diagram:

$$\begin{array}{ccccc} & & & & \mathcal{A}_g \\ & & & \nearrow & \downarrow \\ \operatorname{Spec}(L) & \longrightarrow & \operatorname{Spec} K(A_g) & \longrightarrow & A_g \end{array}$$

**Question 3.1.** *What is the minimal degree $d(g)$ of a field extension $L \supseteq K(A_g)$ such that the map* $\operatorname{Spec}(L) \to A_g$ *lifts to a map to the stack $\mathcal{A}_g$?*

**Known Cases.** Here's what is known.

- $d(1) = 1$
- $d(2) = 2$
- $d(3) = d(5) = 1$ (Fakhruddin in private communication with Vistoli)
- $d(g) \geq 2$ when $g$ is even.
- If $g = 2^s h$ with $h$ odd, then $d(g) \leq 2^{s+3}$.
- $d(g)$ is also always a power of two.

References: Brosnan, Reichstein, Vistoli on "Essential Dimension."

## 4. Brauer group versus cohomological Brauer group (Olsson)

**Background.** $H^2_{et}(X, \mathbb{G}_m)_{\text{tors}}$ is the cohomological Brauer group. The Brauer Group $Br(X)$ is defined as the space of central simple $\mathcal{O}_X$ mod equivalence (reference for a precise definition?). There is an isomorphism $H^2_{et}(X, \mathbb{G}_m)_{\text{tors}} \cong Br(X)$ when $X$ is quasi-projective. However, there exist pathological schemes for which $H^2_{et}(X, \mathbb{G}_m)_{\text{tors}} \ncong Br(X)$.

**Question 4.1.** *Let $X$ be a scheme which is somewhere "between" quasi-projective and pathological. For instance, let $Y$ a projective scheme, $G$ a group whose action on $Y$ is not linearizable, and $X = Y/G$. Or let $X$ be some coarse moduli space. In these cases, is $H^2_{et}(X, \mathbb{G}_m)_{tors} \cong Br(X)$?*

## 5. Foliations and Uniruledness (Kebekus)

**Background.** If $T_X$ is positive, then $X$ contains rational curves. For a more precise statement, recall the theorem of Mehta-Ramanathan which asserts that if $A$ is sufficiently ample, and $C$ is a curve cut out by general elements in the linear system of $A$, then taking the Harder-Narasimhan filtration commutes with restriction to $C$: the Harder-Narasimhan filtration of $T_X$ restricted to $C$ is the restriction of the Harder-Narasimhan filtration of $T_X$ on $X$. In this setup, we say that $C$ is a general complete intersection curve. The Miyoka Uniruledness Criterion says that if $C \subseteq X$ is a general complete intersection curve (see below) and if $\Omega_X|_C$ is not nef, then $X$ is uniruled.

**Question 5.1.** *The condition of $X$ being uniruled is clearly a birational statement. Is there a birational replacement for the existence of a general complete intersection curve $C \subseteq X$ such that $\Omega_X|_C$ not nef?*

Choose $X$ normal, $H$ ample, and consider the associated . Then there is a Harder-Narasimhan filtration:

$$T_X \supseteq \cdots \supseteq F_2 \supseteq F_1 \supseteq 0$$

If $i$ is any number such that the slope of $F_i/F_{i-1}$ is positive, then $F_i$ is a foliation, and the leaves are rationally connected varieites.

**Question 5.2.** *How do these foliations depend on the choice of $H$? See the MRC quotient? Does the filtration "see" the density of rational curves? The long term goal is to prove stability of $T_X$ for $X$ Fano with second betti number equal to* 1.

Reference: Jun Muk Hwang (stability of tangent bundle in small dimensions), Kebekus/Toma/Sola Journal of Algebraic 2007.

## 6. Degenerations of intermediate Jacobians of cubic 3-folds (Alexeev)

**Background.** In principle, for any 1-parameter family of principally polarized abelian varieties (PPAV), there is a unique limit which is a "singular PPAV" which is reduced and which has a nice combinatorial description by a periodic polytopal decomposition (?).

**Question 6.1.** *Describe all degenerations of intermediate Jacobians of cubic 3-folds.*

**Known Cases.** What is known:

- In the case of ordinary Jacobians, the combinatorial data is a "cographic matroid." (See Alexeev for references.)
- Gwena ($\approx$ 2004) described some of these, where the combinatorial data is a matroid $R_{10}$ which is not cographic.
- There is a theory for degenerations of Pryms, which may gave a framework for this...

## 7. Abudnance for boundary divisors on $\overline{M}_{0,n}$ (McKernan)

**Background.** Consider two very hard conjectures

- Abundance Conjecture: If $(K_X + \Delta)$ is KLT and Nef, then it is semi-ample, i.e. some multiple is base-point free. (3.12, p.81 in Kollar-Mori's "Birational geometry of Algebraic Varieties" (CUP, 1998).)

- Let $\overline{M}_{0,n}$ be the moduli space of stable $n$-pointed rational curves. It is conjectured that the cone of curves is spanned by the 1-dimensional boundary strata. Motivated by this, Keel has conjectured that $\overline{M}_{0,n}$ is always a Mori dream space. In particular, Keel's conjecture would imply that every Nef divisor on $\overline{M}_{0,n}$ should be semi-ample. (Roughly, these conjectures say that "all interesting loci on $\overline{M}_{0,n}$ are numerically modular.")

The following question is perhaps an easier version related to the above hard conjectures:

**Question 7.1.** *Let $M = \overline{M}_{0,n}$ and consider $K_M + \Delta$ where $\Delta$ is some collection of boundary divisors with nonnegative coefficents less than 1 (this implies that $(M, \Delta)$ is KLT). If $K_M + \Delta$ is nef, then is it semi-ample?*

Also, it's not known exactly which conditions on $\Delta$ guarantee that $K_M + \Delta$ is nef. This would be interesting for divisors $\Delta$ which are collections of boundary divisors. (Note that if $\Delta$ is the total boundary $\partial\overline{M}_{0,n}$ then this is known to be okay. See for example Hassett, Hyeon-Lee, Smyth.)

Other references: Keel-McKernan and Castravet-Tevelev.

## 8. Mobility thresholds (Corti)

**Background.** Let $X$ have terminal singularities, $D$ a $\mathbb{Q}$-Cartier divisor, and $x \in X$. Then

$$\delta(D, x) := \sup_{f:Y \to X} \frac{1}{a} \sup\{\beta \geq 0 | f^*D - \beta E \text{ is mobile.}\}$$

where $f$ is an extremal divisorial contraction with exceptional divisor $E$, $f(E) = x$, and $a$ is the discrepancy of $f$. We refer to $\delta(D, x)$ as the mobility threshold of $D$ at $x$.

**Question 8.1.** *Compute all mobility thresholds for $X$ a quartic 3-fold.*

**Known Cases.** It is known that if $X$ is a smooth quartic 3-fold and $D = \mathcal{O}(1) = -K_X$, then $\delta(D, x) \leq 1$ (Manin-Iskovskikh). When is $\delta(D, x) > 1$? In this case, $X$ can have different models as a Mori fiber space. Also, the invariant is interesting in general because it controls some of the birational geometry of $X$. Namely, it involves the Noether-Fano-Iskovskikh inequality. If $f : X \to Y$ is a nontrivial birational map, then $\delta(-K, x) > 1$ for some $x$.

One interesting case is $X$ a 3-fold and $x$ some sinularity which is slightly more complicated than an ordinary node. E.g. $x^2 + y^2 + z^3 + t^3$. Or perhaps a quartic 3-fold with terminal singularities.

**Conjecture (believed to be false...)** If $X \to S$ and $X' \to S$ are two families of a $\mathbb{Q}$-Fano 3-folds, then the set of pairs $\{(s, t) | X_s \text{ is birational to } X'_t\}$ is a constructible set. (Even more likely to be false for cubic 4-folds. This is because there is a countable union of divisors in the moduli space of cubic 3-folds which "seem to be rational.")

## 9. Cohomology of vector bundles on $\mathbb{P}^a \times \mathbb{P}^b$ (Eisenbud)

**Background.** Let $\mathcal{E}$ a vector bundle on $\mathbb{P}^n$ and consider the cohomology table $\gamma(\mathcal{E})$ defined by $\gamma_{i,d} = h^i\mathcal{E}(d)$. It turns out that this is a positive rational combination of cohomology tables of "supernatural bundles." (See for example Eisenbud-Schreyer.)

**Definition 9.1.** A vector bundle $\mathcal{F}$ is supernatural if for all $d$ there exists at most one $i$ such that $h^o(\mathcal{E}(d)) \neq 0$ and if $\chi(\mathcal{E}(d))$ has $n$ distinct roots.

**Question 9.2.** *Is there a similar decomposition theorem for vector bundles on $\mathbb{P}^a \times \mathbb{P}^b$? Even $\mathbb{P}^1 \times \mathbb{P}^1$ would be an interesting case.*

## 10. Complete curves in $C \times \cdots \times C \setminus \Delta$ (Faber)

Let $C$ be a smooth projective curve.

**Question 10.1.** *For which curves $C$ does $C \times C \times C \setminus$ all diagonals contain a complete curve?*

**Known Cases.** If $C$ is elliptic, then you can do this for $C \times \cdots \times C$ for any number of factors. You use the group law to translate the small diagonal of $C$ by distinct points in different factors. If $g(C) \geq 2$ then $C \times C \to Jac$ mapping by $(P, Q) \mapsto P - Q$ gives a map which sends the diagonal to a single point of the Jacboian. So you can pull back a complete curve in $C \times C \setminus \Delta$.

**Question 10.2.** *Does there exist a map contracting the diagonal $\Delta$.*

References: (Keel, from Annals).

## 11. Section rings (Reid)

**Background.** Let $A$ be a finitely generated graded ring with $A_0 = \mathbb{C}$. Then one can consider the Hilbert function $HF_A(n) := \dim_C A_n$. If $A = \oplus_{n\geq 0} H^0(nK_S)$ is the canonical ring of a surface of general type, then there is a famous formula for the Hilbert function which is due to Kodaira (see [Kod68]), essentially Riemann–Roch plus Kodaira vanishing. If $A$ is the canonical ring of a 3-fold of general type, then the periodic polynomial for the Hilbert function reduces to a nice RR polynomial plus a periodic contribution arising from the canonical singularities of index greater than 1. Moreover, the periodic contributions can be calculated in terms of equivariant Lefschetz formula (now "orbifold Riemann–Roch"), and are expressed as Dedekind sums (now "ice cream"). See [Rei87, Chap. III].

**Question 11.1.** *Give some geometric meaning to the periodic contributions to the Hilbert function for the canonical ring of a 4-fold of general type.*

Since the canonical model of a 4-fold of general type has only 1-dimensional locus of points of index $\geq 1$, the"periodic polynomia" Hilbert function has periodic contributions that grow at most linearly. If the canonical model is a smooth DM stack (i.e., has only orbifold canonical points), the question is concerned with orbifold Riemann–Roch, and one could try to reduce it to equivariant RR following [Rei87]; even under this simplifying assumption, the question of localising the resulting "nonisolated Dedekind sums" is open. In general, 4-fold canonical singularities are thought to be an intractable class, but that does not necessarily mean that their plurigenus contributions are beyond the resources of rational thought.

It is instructive to do simpler examples in low dimensional cases; for example, how to understand the Hilbert series of a curve such as $C_{14} \subset \mathbb{P}(2, 3, 5)$? The graded ring is $A = k[x, y, z]/f_{14}$, where maybe

$$f_{14} = x^7 + xy^4 + x^2z^2 - y^3z + \text{other terms.}$$

Spec$(A)$, which is the affine cone over $C_{14}$, is nonsingular along the $y$-axis outside the origin, but has a line of cusps $(x^2 - y^3 = 0)$ along the $z$-axis. Despite this, the scheme $C_{14} = \operatorname{Proj} A$ (= coarse moduli space) is a nonsingular elliptic curve: it has orbifold point $\frac{1}{3}(2)$ at $P = (0,1,0)$, with either $x$ or $z$ as orbinate, and orbicusp $\frac{1}{5}(2,3)[x^2+y^3]$ at $Q = (0,0,1)$. The orbinate at $Q$ would be $x/y$ (normalising the line of cusps), but it is not present in the ring of $C$.

You can view this as $C_{14}$ = elliptic curve, marked with $\mathbb{Q}$-divisor $\frac{2}{3}P + \frac{4}{5}Q - R$ (with linear equivalence $2P + 2Q \sim 4R$), but with a taboo against the orbinate at $Q$. Its Hilbert series is

$$(1 + t + t^2 + t^4) + \frac{7}{15}\frac{t}{(1-t)^2} + \frac{(-2/3)t - (1/3)t^2}{1-t^3} \\ + \frac{(-4/5)t - (3/5)t^2 - (2/5)t^3 - (1/5)t^4}{1-t^5} - \frac{t^4}{1-t^5} \\ = \frac{1-t^{14}}{\prod_{a\in[2,3,5]}(1-t^a)}.$$

Figure out what each term means!

## References


[Kod68] Kunihiko Kodaira, *Pluricanonical systems on algebraic surfaces of general type*, J. Math. Soc. Japan **20** (1968), 170–192. MR0224613 (37 #212) ↑11

[Rei87] Miles Reid, *Young person's guide to canonical singularities*, Algebraic geometry, Bowdoin, 1985 (Brunswick, Maine, 1985), Proc. Sympos. Pure Math., vol. 46, Amer. Math. Soc., Providence, RI, 1987, pp. 345–414. MR927963 (89b:14016) ↑11, 11