Introduction to Symplectic Geometry and Topology

Exercises for Lecture 1

Exercise 1. Check that the vector field $X_H = -J\nabla H$ is indeed the Hamiltonian vector field for a smooth function H on $(\mathbb{R}^{2n}, \omega_0)$.

Exercise 2. (*Good for getting a feel for the various types of subspaces.*) Consider V with the standard symplectic form $\sum_{i=1}^{n} e_i^* \wedge f_i^*$ with respect to basis $B = \{e_1, \dots e_n, f_1, \dots f_n\}$. Describe isotropic, coisotropic, Lagrangian and symplectic subspaces spanned by subsets of B.

Exercise 3. *(Underlies symplectic reduction in Lecture 3.)* Restricting a symplectic form ω to a subspace $W \subset (V, \omega)$ yields a degenerate symplectic form unless W is symplectic. However, show ω always induces a well-defined symplectic form on the quotient space $W/(W \cap W^{\perp})$.

Exercise 4. Any vector space V equipped with a skew-symmetric form Ω decomposes as a direct sum $(U \oplus V_1 \oplus \dots V_n, \omega_1 \oplus \dots \omega_n)$ where each (V_i, ω_i) is a of 2-dimensional symplectic vector spaces. In particular, there exists a basis $e_1, \dots e_n, f_1, \dots f_n$ with respect to which $\omega = e^* \wedge f^* = \sum_{i=1}^{n} e_i^* \wedge f_i^*$ (The half of Witt's theorem pertaining to skew-symmetric forms).

Exercise 5. *(Important for the Lagrangian neighborhood theorem.)* Consider a Lagrangian subspace $L \subset (V, \omega)$ with basis $\{u_1, \dots u_n\}$. Let L^* be the dual space $u_i^* = \omega(u_i, \cdot)$. Let ω_0 be the unique symplectic form such that $\omega_0(u_i, u_j^*) = u_j^*(u_i)$. Check that the linear map Φ that sends u_i to itself and v_i to u_i^* defines an isomorphism $(V, \omega) \cong (L \oplus L^*, \omega_0)$.

Exercise 6. *(Do this one for sure.)* Show that a linear map $\Psi : V \to V$ is a symplectomorphism of (V, ω) if and only if its graph $\{(v, \Psi(v)) | v \in V\}$ is a Lagrangian subspace of $(V \oplus V, -\omega \oplus \omega)$.

Exercise 7. *(Basic.)* Verify that a matrix A represents a symplectomorphism if and only if $A^T J_0 A = J_0$ and that the set of all such A forms a group.

Exercise 8. Verify that

$$\mathrm{U}(n) = \mathrm{Sp}(2n) \cap \mathrm{GL}(n, \mathbb{C}) = \mathrm{Sp}(2n) \cap \mathrm{O}(n) = \mathrm{O}(2n) \cap \mathrm{GL}(n, \mathbb{C})$$